Calculus 2502A - Advanced Calculus I
Fall 2014
§12.5: Intersection of two planes

Martin Frankland

September 19, 2014

Example 1. Find an equation for the line of intersection of the two planes defined by the equations $x + y + z = 3$ and $2x - y + 3z = 0$.

Geometric solution. The first plane has normal vector $\vec{n}_1 = (1, 1, 1)$ while the second plane has normal vector $\vec{n}_2 = (2, -1, 3)$. Note that they are not parallel, and thus they intersect in a line.

Since the line of intersection is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$, we can take its direction vector to be:

$$\begin{aligned}
\vec{v} &= \vec{n}_1 \times \vec{n}_2 \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} \\
&= \vec{i} \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\
&= \vec{i}(3 - (-1)) - \vec{j}(3 - 2) + \vec{k}(-1 - 2) \\
&= 4\vec{i} - \vec{j} - 3\vec{k} \\
&= (4, -1, -3).
\end{aligned}$$

It remains to find a point on the line. Let us find the point on the line with $z = 0$, in which case the equations of the two planes become:

$$\begin{cases} x + y = 3 \\ 2x - y = 0 \end{cases}$$

which has the (unique) solution $x = 1$, $y = 2$, yielding the point $(1, 2, 0)$. The line of intersection of the two planes has as vector equation:

$$\boxed{\vec{r}(t) = (1, 2, 0) + t(4, -1, -3)} \text{ for } t \in \mathbb{R}.$$

Alternately, the parametric equations of the line are:

$$\boxed{\begin{cases} x(t) = 1 + 4t \\ y(t) = 2 - t \\ z(t) = -3t \end{cases}}$$

for $t \in \mathbb{R}$.

Algebraic solution. The intersection of the two planes is the solution set of the system of two equations:

$$\begin{cases} x + y + z = 3 \\ 2x - y + 3z = 0 \end{cases}$$

$$\sim \begin{cases} x + y + z = 3 \\ \quad -3y + z = -6 \end{cases}$$

$$\sim \begin{cases} x + y + z = 3 \\ \quad y - \frac{1}{3}z = 2 \end{cases}$$

$$\sim \begin{cases} x \quad + \frac{4}{3}z = 1 \\ \quad y - \frac{1}{3}z = 2. \end{cases}$$

Using $z = t$ as parameter, the second equation yields $y = 2 + \frac{1}{3}t$ and the first equation yields $x = 1 - \frac{4}{3}t$. Therefore, the parametric equations of the line are:

$$\boxed{\begin{cases} x(t) = 1 - \frac{4}{3}t \\ y(t) = 2 + \frac{1}{3}t \\ z(t) = t \end{cases}}$$

for $t \in \mathbb{R}$.

Remark 2. Fraction-haters may as well use $\frac{1}{3}z = s$ as parameter, yielding the parametric equations:

$$\boxed{\begin{cases} x(s) = 1 - 4s \\ y(s) = 2 + s \\ z(s) = 3s \end{cases}}$$

for $s \in \mathbb{R}$.