Calculus 2502A - Advanced Calculus I
Fall 2014
§13.3: Osculating and normal planes

Martin Frankland

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Example 1. Consider the curve parametrized by

$$\vec{r}(t) = \left(t+3, t^2-1, t^3-4\right)$$

for $t \in \mathbb{R}$.
a) Find an equation for the normal plane to the curve at the point $(5, 3, 4)$.
b) Find an equation for the osculating plane of the curve at the point $(5, 3, 4)$.
c) Find all points (if any) on the curve where the osculating plane is parallel to the plane defined by the equation $3z = 7 - 9x - 9y$.

Solution. **a)** The tangent vector is

$$\vec{r}\,'(t) = \left(1, 2t, 3t^2\right).$$

The point $\vec{r}(t) = (5, 3, 4)$ corresponds to the (unique) value of the parameter $t = 2$, for which the tangent vector is

$$\vec{r}\,'(2) = (1, 4, 12).$$

The normal plane at the point $(5, 3, 4)$ has equation:

$$\boxed{(x-5) + 4(y-3) + 12(z-4) = 0}$$

or equivalently:

$$\boxed{x + 4y + 12z = 65}.$$

b) The second derivative is

$$\vec{r}\,''(t) = (0, 2, 6t).$$

Note is particular that $\vec{r}\,''(t)$ is never parallel to $\vec{r}\,'(t)$, and so the osculating plane is defined at every point of the curve. At the point $\vec{r}(2) = (5, 3, 4)$, we have

$$\vec{r}\,''(2) = (0, 2, 12).$$

The osculating plane is parallel to both $\vec{r}'(2)$ and $\vec{r}''(2)$, and so we can take as its normal vector:

$$\begin{aligned}
\vec{r}'(2) \times \vec{r}''(2) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 12 \\ 0 & 2 & 12 \end{vmatrix} \\
&= \vec{i}\begin{vmatrix} 4 & 12 \\ 2 & 12 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 12 \\ 0 & 12 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} \\
&= \vec{i}(48-24) - \vec{j}(12-0) + \vec{k}(2-0) \\
&= (24, -12, 2)
\end{aligned}$$

or we may as well take the scalar multiple $(12, -6, 1)$. The osculating plane at the point $(5, 3, 4)$ has equation:

$$\boxed{12(x-5) - 6(y-3) + (z-4) = 0}$$

or equivalently:

$$\boxed{12x - 6y + z = 46}.$$

c) At any point $\vec{r}(t)$ on the curve, the osculating plane has as its normal vector:

$$\begin{aligned}
\vec{r}'(t) \times \vec{r}''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} \\
&= \vec{i}\begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} \\
&= \vec{i}(12t^2 - 6t^2) - \vec{j}(6t-0) + \vec{k}(2-0) \\
&= (6t^2, -6t, 2)
\end{aligned}$$

or we may as well take $(3t^2, -3t, 1)$. The given plane has normal vector $(9, 9, 3)$. We are looking for all values of t such that $(3t^2, -3t, 1)$ is parallel to $(9, 9, 3)$, i.e.,

$$(3t^2, -3t, 1) = c(9, 9, 3)$$

for some scalar $c \neq 0$. This condition is equivalent to

$$(3t^2, -3t, 1) = (3, 3, 1)$$

which holds if and only if $t = -1$ holds. Thus, the only point on the curve satisfying the condition is

$$\vec{r}(-1) = \boxed{(2, 0, -5)}.$$