Calculus 2502A - Advanced Calculus I
Fall 2014
§13.4: Tangential and normal acceleration

Martin Frankland

October 17, 2014

Notation 1. Denote by:

- $\vec{r}(t)$ the position of a particle at time t.
- $\vec{v}(t) := \vec{r}\,'(t)$ the velocity.
- $v(t) := |\vec{v}(t)|$ the speed.
- $\vec{a}(t) := \vec{r}\,''(t)$ the acceleration.
- $\vec{T}(t) := \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ the unit tangent vector, whenever $\vec{r}\,'(t) \neq \vec{0}$.
- $\vec{N}(t) := \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$ the principal unit normal vector, whenever $\vec{T}'(t) \neq \vec{0}$.
- $\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}$ the curvature.

The acceleration vector $\vec{a}(t)$ is always in the osculating plane, spanned by $\vec{T}(t)$ and $\vec{N}(t)$.

Notation 2. Write the acceleration vector as

$$\vec{a}(t) = a_T\,\vec{T}(t) + a_N\,\vec{N}(t)$$

where a_T denotes the **tangential component** of $\vec{a}(t)$ and a_N denotes the **normal component** of $\vec{a}(t)$.
Note that $a_N \geq 0$ holds for all t, by our choice of the normal vector $\vec{N}(t)$.

Proposition 3. *The tangential component of the acceleration is given by*

$$a_T = v' = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|}.$$

The normal component of the acceleration is given by

$$a_N = \kappa v^2 = \frac{|\vec{v} \times \vec{a}|}{v} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|}.$$

Proof. Recall that the unit tangent vector is defined by the equation

$$\vec{r}\,' = v\vec{T}$$

and the principal unit normal vector is defined by the equation

$$\vec{T}' = |\vec{T}'|\vec{N}.$$

Acceleration is given by

$$\begin{aligned} \vec{a} &= \vec{r}\,'' \\ &= (v\vec{T})' \\ &= v'\vec{T} + v\vec{T}' \\ &= v'\vec{T} + v|\vec{T}'|\vec{N}. \end{aligned} \tag{1}$$

Now recall that curvature is given by

$$\kappa = \frac{|\vec{T}'|}{|r'|} = \frac{|\vec{T}'|}{v}$$

and substituting $|\vec{T}'| = \kappa v$ into equation (1) yields

$$\vec{a} = v'\vec{T} + \kappa v^2\vec{N}.$$

Dotting the acceleration with the unit tangent vector yields

$$\begin{aligned} \vec{T} \cdot \vec{a} &= \vec{T} \cdot \left(a_T\,\vec{T} + a_N\,\vec{N}\right) \\ &= a_T\,\vec{T} \cdot \vec{T} + a_N\,\vec{T} \cdot \vec{N} \\ &= a_T \end{aligned}$$

from which we obtain

$$a_T = \frac{\vec{v}}{v} \cdot \vec{a} = \frac{\vec{v} \cdot \vec{a}}{v}.$$

Crossing the acceleration with the unit tangent vector yields

$$\begin{aligned} \vec{T} \times \vec{a} &= \vec{T} \times \left(a_T\,\vec{T} + a_N\,\vec{N}\right) \\ &= a_T\,\vec{T} \times \vec{T} + a_N\,\vec{T} \times \vec{N} \\ &= a_N\,\vec{T} \times \vec{N} \\ \Rightarrow |\vec{T} \times \vec{a}| &= |a_N\,\vec{T} \times \vec{N}| \\ &= |a_N||\vec{T} \times \vec{N}| \\ &= a_N \end{aligned}$$

from which we obtain

$$a_N = |\frac{\vec{v}}{v} \times \vec{a}| = \frac{|\vec{v} \times \vec{a}|}{v}.$$

□

Proposition 4. *The following conditions for a parametrized curve $\vec{r}(t)$ are equivalent (assuming $\vec{r}$ is twice differentiable and $\vec{r}\,'(t)$ is never zero).*

1. *The acceleration is always orthogonal to the velocity, i.e., $\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$ holds for all t.*

2. *The tangential component a_T of the acceleration is identically zero, i.e., $a_T = 0$ holds for all t.*

3. *The speed $|\vec{r}\,'(t)|$ is constant.*

Proof. $(1 \Rightarrow 2)$ By Proposition 3, we have

$$a_T = \frac{\vec{r}\,' \cdot \vec{r}\,''}{|\vec{r}\,'|} \equiv 0.$$

$(2 \Rightarrow 3)$ The speed function $v(t)$ is differentiable and has derivative $v' = a_T \equiv 0$, so that v must be constant. (Recall that the domain of $\vec{r}$ is an interval.)
$(3 \Rightarrow 1)$ The function $|\vec{r}\,'|^2 = \vec{r}\,' \cdot \vec{r}\,'$ is constant, so that its derivative is identically zero:

$$\begin{aligned} 0 &= (\vec{r}\,' \cdot \vec{r}\,')' \\ &= \vec{r}\,'' \cdot \vec{r}\,' + \vec{r}\,' \cdot \vec{r}\,'' \\ &= 2\vec{r}\,' \cdot \vec{r}\,''. \end{aligned}$$

□

Example 5. Consider a particle moving along a circle of radius R increasingly fast, with position function

$$\vec{r}(t) = \left(R\cos t^2, R\sin t^2\right)$$

for $t \geq 0$. Let us find the tangential component a_T and normal component a_N of the acceleration.
The velocity is

$$\begin{aligned} \vec{r}\,'(t) &= R\left(-2t\sin t^2, 2t\cos t^2\right) \\ &= 2Rt\left(-\sin t^2, \cos t^2\right) \end{aligned}$$

whose magnitude is

$$|\vec{r}\,'(t)| = 2Rt.$$

The acceleration is

$$\begin{aligned} \vec{r}\,''(t) &= 2R\left(-\sin t^2, \cos t^2\right) + 2Rt\left(-2t\cos t^2, -2t\sin t^2\right) \\ &= 2R\left(-\sin t^2, \cos t^2\right) - 4Rt^2\left(\cos t^2, \sin t^2\right). \end{aligned}$$

Then we have:

$$\begin{aligned}
\vec{r}'(t) \cdot \vec{r}''(t) &= 2Rt\left(-\sin t^2, \cos t^2\right) \cdot \left(2R\left(-\sin t^2, \cos t^2\right) - 4Rt^2\left(\cos t^2, \sin t^2\right)\right) \\
&= 4R^2t\left(-\sin t^2, \cos t^2\right) \cdot \left(-\sin t^2, \cos t^2\right) - 8R^2t^3\left(-\sin t^2, \cos t^2\right) \cdot \left(\cos t^2, \sin t^2\right) \\
&= 4R^2t(1) - 8R^2t^3(0) \\
&= 4R^2t
\end{aligned}$$

and therefore the tangential component of the acceleration is

$$\begin{aligned}
a_T &= \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|} \\
&= \frac{4R^2t}{2Rt} \\
&= \boxed{2R}.
\end{aligned}$$

We also have (viewing $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$):

$$\begin{aligned}
\vec{r}'(t) \times \vec{r}''(t) &= 2Rt\left(-\sin t^2, \cos t^2, 0\right) \times \left(2R\left(-\sin t^2, \cos t^2, 0\right) - 4Rt^2\left(\cos t^2, \sin t^2, 0\right)\right) \\
&= -8R^2t^3\left(-\sin t^2, \cos t^2, 0\right) \times \left(\cos t^2, \sin t^2, 0\right) \\
&= -8R^2t^3\left(-\sin^2 t^2\, \vec{i} \times \vec{j} + \cos^2 t^2\, \vec{j} \times \vec{i}\right) \\
&= -8R^2t^3\left(-\sin^2 t^2 - \cos^2 t^2\right)\vec{k} \\
&= 8R^2t^3\vec{k}
\end{aligned}$$

and therefore the normal component of the acceleration is

$$\begin{aligned}
a_N &= \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|} \\
&= \frac{8R^2t^3}{2Rt} \\
&= \boxed{4Rt^2}.
\end{aligned}$$

Remark 6. We can explicitly verify Proposition 3 in Example 5.
The speed function is $v(t) = 2Rt$ and its derivative is $v'(t) = 2R$, so that the equality $\boxed{a_T = v'}$ holds indeed.
The curvature of a circle of radius R is the constant function $\kappa(t) = \frac{1}{R}$. Thus, we have

$$\begin{aligned}
\kappa v^2 &= \frac{1}{R}(2Rt)^2 \\
&= \frac{4R^2t^2}{R} \\
&= 4Rt^2
\end{aligned}$$

so that the equality $\boxed{a_N = \kappa v^2}$ holds indeed.