Calculus 2502A - Advanced Calculus I
Fall 2014
§14.3: Clairaut's theorem

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Theorem 1 (Clairaut's theorem). *Let $f\colon D \to \mathbb{R}$ be a function with domain $D \subseteq \mathbb{R}^2$, and let (a,b) be an interior point of D. If the second partial derivatives f_{xy} and f_{yx} exist and are continuous in a neighborhood of (a,b), then they satisfy $f_{xy}(a,b) = f_{yx}(a,b)$.*

The following (non-)example illustrates why the assumptions of the theorem are important.

Example 2 (# 14.3.101). Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

At all points $(x,y) \neq (0,0)$, the partial derivative f_x is given by:

$$\begin{aligned} f_x(x,y) &= \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} \\ &= \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}. \end{aligned}$$

Given the equality $f(x,0) = 0$ for all x, we have $f_x(0,0) = 0$ and therefore:

$$f_x(x,y) = \begin{cases} \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Likewise, the partial derivative f_y is given by:

$$f_y(x,y) = \begin{cases} \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

To compute the mixed partial derivatives at the origin, consider:

$$f_x(0,y) = \frac{-y^5}{y^4} = -y$$

for all y, which implies $\boxed{f_{xy}(0,0) = -1}$. Likewise, we have:

$$f_y(x,0) = \frac{x^5}{x^4} = x$$

for all y, which implies $\boxed{f_{yx}(0,0) = 1}$.

Does this contradict Clairaut's theorem? No: f does not satisfy the assumptions of the theorem. The second partial derivatives f_{xy} and f_{yx} (and f_{xx} and f_{yy} for that matter) exist on all of $\mathbb{R}^2$, in particular in a neighborhood of $(0,0)$. However, f_{xy} and f_{yx} are not continuous at $(0,0)$.