Calculus 2502A - Advanced Calculus I
Fall 2014
§14.7: Global minima and maxima

Martin Frankland

November 19, 2014

In these notes, we discuss the problem of finding the global (or absolute) minima and maxima of a function.

1 One-dimensional case

Assume for now that $f \colon D \to \mathbb{R}$ is a function of one variable, with domain $D \subseteq \mathbb{R}$. An important tool for finding global extrema of a function is the following theorem.

Theorem 1.1 (Extreme value theorem)**.** *Let $f \colon [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$. Then f attains a (global) minimum and a (global) maximum on $[a, b]$.*

Remark 1.2. The assumption that the interval be closed is important. For example, consider the function $f \colon (0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$. Then f does *not* have a global maximum on $(0, 1]$. Here the theorem does not apply, because the interval $(0, 1]$ is not closed.

Remark 1.3. The assumption that the interval be bounded, i.e., a and b are finite numbers, is also important. For example, consider the function $f \colon [0, +\infty) \to \mathbb{R}$ defined by $f(x) = x$. Then f does *not* have a global maximum on $[0, +\infty)$. Here the theorem does not apply, because the interval $[0, +\infty)$ is not bounded – though it **is** closed.

Upshot: The extreme value theorem provides a method for finding the (global) extrema of a function $f \colon [a, b] \to \mathbb{R}$.

1. Look for critical points of f in the interval (a, b), i.e., the interior of the domain of f. Evaluate f at the critical points.

2. Look at the values of f at the endpoints $x = a$ and $x = b$, i.e., the boundary of the domain of f.

Example 1.4. Let $f\colon [0,3] \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{2x+1}{x^2+1}.$$

Find the maximal and minimal values of f, and where they occur.

Solution. Let us find the critical points of f:

$$\begin{aligned} f'(x) &= \frac{2(x^2+1) - (2x+1)(2x)}{(x^2+1)^2} \\ &= \frac{2x^2+2-4x^2-2x}{(x^2+1)^2} \\ &= \frac{-2x^2-2x+2}{(x^2+1)^2}. \end{aligned}$$

Setting $f'(x) = 0$, we obtain the equation

$$\begin{aligned} & -2x^2 - 2x + 2 = 0 \\ \Leftrightarrow\, & x^2 + x - 1 = 0 \\ \Leftrightarrow\, & x = \frac{-1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

so that $x = \frac{\sqrt{5}-1}{2}$ is the only critical point of f, since the domain of f is $[0,3]$. There, the value of f is

$$\begin{aligned} f\left(\frac{\sqrt{5}-1}{2}\right) &= \frac{2\left(\frac{\sqrt{5}-1}{2}\right)+1}{\left(\frac{\sqrt{5}-1}{2}\right)^2+1} \\ &= \frac{\sqrt{5}}{\frac{3-\sqrt{5}}{2}+1} \\ &= \frac{\sqrt{5}}{\frac{5-\sqrt{5}}{2}} \\ &= \frac{2\sqrt{5}}{5-\sqrt{5}} \\ &= \frac{10\sqrt{5}+10}{25-5} \\ &= \frac{\sqrt{5}+1}{2} \approx 1.62. \end{aligned}$$

At the endpoints $x = 0$ and $x = 3$, the values of f are

$$f(0) = \frac{1}{1} = 1$$
$$f(3) = \frac{6+1}{9+1} = \frac{7}{10}.$$

Therefore the (global) **minimum** of f is $\boxed{\frac{7}{10}}$, which occurs at $\boxed{x = 3}$.

The (global) **maximum** of f is $\boxed{\frac{\sqrt{5}+1}{2}}$, which occurs at $\boxed{x = \frac{\sqrt{5}-1}{2}}$.

Note that $x = 0$ is a local minimum of f.

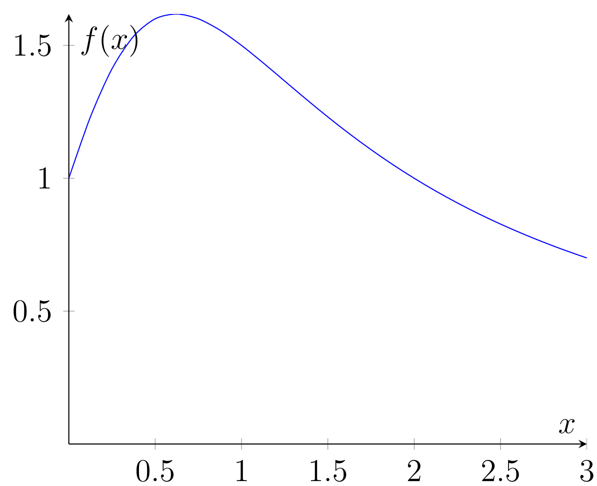

Figure 1: Graph of f from Example 1.4.

2 In higher dimension

Theorem 1.1 has an analogue in higher dimension.

Definition 2.1. A subset $D \subseteq \mathbb{R}^n$ is called:

1. **closed** if D contains all its limit points.
2. **bounded** if D is contained within some disk (of finite radius).

Here are examples of subsets $D \subseteq \mathbb{R}^2$ illustrating these two notions.

Closed and bounded.

- The disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq c^2\}$ of radius c.
- The rectangle $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$.
- The circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = c^2\}$ of radius c.
- The line segment $\{(x, y) \in \mathbb{R}^2 \mid x + y = 5, 0 \leq x \leq 5\}$.

Closed but NOT bounded.

- The first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$.
- The upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$.
- The infinite horizontal strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$.
- The x-axis $\{(x, y) \in \mathbb{R}^2 \mid y = 0\}$.
- More generally, any line $\{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$.
- All of $\mathbb{R}^2$.

Bounded but NOT closed.

- The "open" disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < c^2\}$ of radius c.
- The punctured disk $\{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq c^2\}$ of radius c, missing its center.
- The rectangle $\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 2, 0 \leq y \leq 1\}$ missing its left edge.
- The line segment $\{(x, y) \in \mathbb{R}^2 \mid x + y = 5, 0 \leq x < 5\}$ missing its endpoint $(5, 0)$.

NEITHER closed NOR bounded.

- The "open" first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$.
- The first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y > 0\}$ missing the half-x-axis.
- The "open" upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$.
- The infinite strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y < 1\}$ missing its upper edge.
- The half x-axis $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y = 0\}$ missing its endpoint $(0, 0)$.

Exercise 2.2. Prove the statements above, about the given subsets $D \subseteq \mathbb{R}^2$.

Theorem 2.3 (Extreme value theorem)**.** *Let $f : D \to \mathbb{R}$ be a continuous function with domain $D \subseteq \mathbb{R}^n$. Assume D is* closed *and* bounded*. Then f attains a (global) minimum and a (global) maximum on D.*

Upshot: The extreme value theorem provides a method for finding the (global) extrema of a function $f : D \to \mathbb{R}$ if the domain D is closed and bounded.

1. Look for critical points of f in the interior of the domain D. Evaluate f at each critical point.
2. Study the values of f on the boundary of the domain D.

The idea is exactly as in the one-dimensional case, although the implementation is slightly more complicated.

Example 2.4. Let $f : D \to \mathbb{R}$ be the function defined by

$$f(x, y) = x^2 - 2xy + 2y$$

having domain the rectangle $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$. Find the minimal and maximal values of f, and where they occur.

Solution. Let us find the critical points of f inside D:

$$f_x(x, y) = 2x - 2y$$
$$f_y(x, y) = -2x + 2.$$

Setting $\nabla f(x, y) = \vec{0}$, we obtain the system of equations

$$\begin{cases} 2x - 2y &= 0 \\ -2x + 2 &= 0 \end{cases}$$

whose only solution is $(x, y) = (1, 1)$. Note that the point $(1, 1)$ is inside D. There, the value of f is

$$f(1, 1) = 1 - 2 + 2 = 1.$$

The boundary of D has four sides.

Left side $x = 0$. The function there is

$$f(0, y) = 2y$$

which reaches a minimum of 0 at $y = 0$ and a maximum of 4 at $y = 2$.

Right side $x = 3$. The function there is

$$f(3, y) = 9 - 6y + 2y = 9 - 4y$$

which reaches a minimum of 1 at $y = 2$ and a maximum of 9 at $y = 0$.

Bottom side $y = 0$. The function there is

$$f(x, 0) = x^2$$

which reaches a minimum of 0 at $x = 0$ and a maximum of 9 at $x = 3$.

Top side $y = 2$. The function there is

$$f(x, 2) = x^2 - 4x + 4$$

which has one critical point at $x = 2$, with value $f(2, 2) = 0$. At the endpoints, the function has values:

$$f(0, 2) = 4$$
$$f(3, 2) = 1.$$

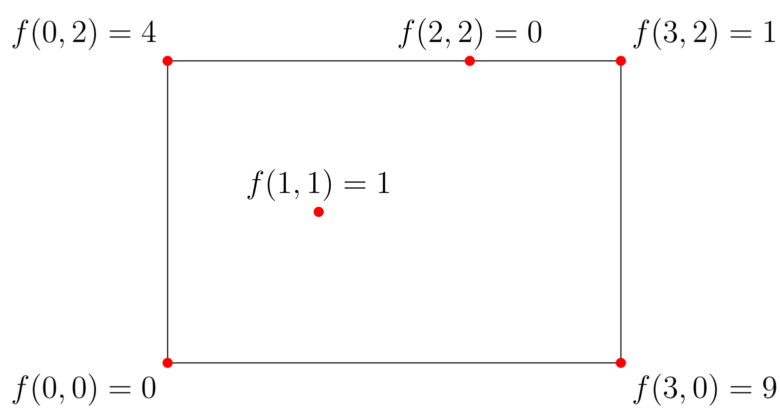

Figure 2: Certain values of f.

Putting all the information together, we conclude the following (see Figure 2).

The (global) **minimum** of f is $\boxed{0}$, which occurs at $\boxed{(x, y) = (0, 0)}$ and $\boxed{(2, 2)}$.

The (global) **maximum** of f is $\boxed{9}$, which occurs at $\boxed{(x, y) = (3, 0)}$.

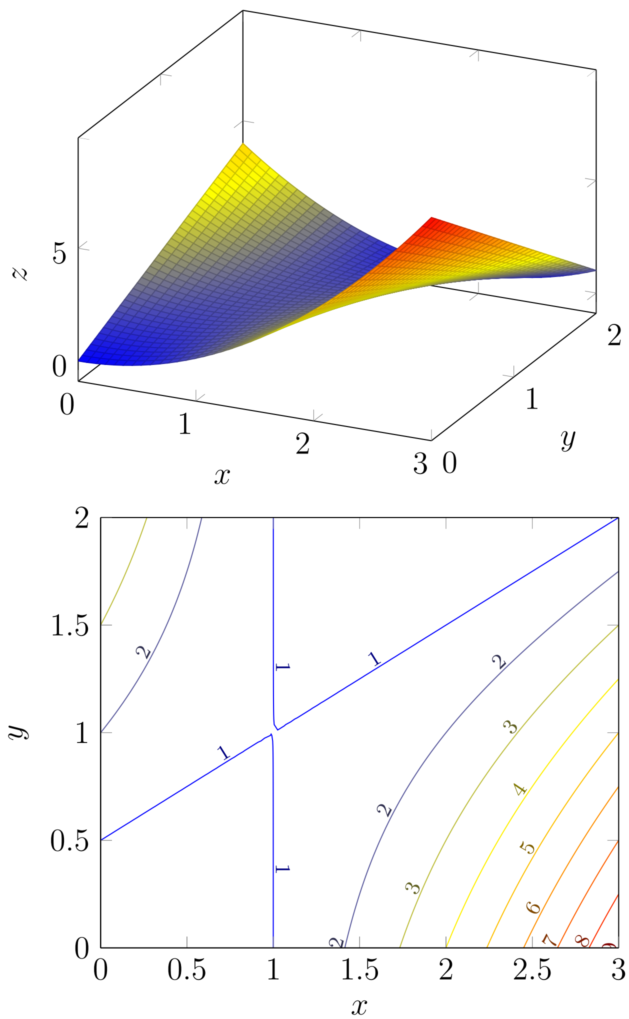

Figure 3: Graph and contour plot of the function f in Example 2.4.

3 Unbounded domains

When the domain D of the function f is not bounded (or not closed), we can still use the extreme value theorem with the following strategy.

1. Restrict the function to some closed and bounded subset D'.
2. Analyze what happens in D' and outside of D'.

Example 3.1. Find the distance from the plane $2x - y + z = 3$ to the origin.

Remark 3.2. The problem is easily solved using projections, as seen in Chapter 12. However, the method described below also works for surfaces other than planes.

Solution. The plane can be expressed by $z = 3 - 2x + y$, in other words, using x and y as parameters. The distance squared from the point $(x, y, 3 - 2x + y)$ to the origin is

$$f(x,y) = x^2 + y^2 + (3 - 2x + y)^2$$

which we want to minimize. Note that the domain of f is $D = \mathbb{R}^2$. Let us find the critical points of f:

$$\begin{aligned} f_x(x,y) &= 2x + 2(3 - 2x + y)(-2) \\ &= 10x - 4y - 12 \\ f_y(x,y) &= 2y + 2(3 - 2x + y)(1) \\ &= -4x + 4y + 6. \end{aligned}$$

Setting $\nabla f(x) = \vec{0}$, we obtain the system of equations

$$\begin{cases} 10x - 4y &= 12 \\ -4x + 4y &= -6 \end{cases}$$

whose only solution is $(x,y) = (1, -\frac{1}{2})$. Geometrically, this unique critical point must a minimum, and so the minimum value of f is

$$\begin{aligned} f(1, -\frac{1}{2}) &= (1)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= 1 + \frac{1}{4} + \frac{1}{4} \\ &= \frac{3}{2}. \end{aligned}$$

The distance from the plane to the origin is therefore $\boxed{\sqrt{\frac{3}{2}}}$.

How to show rigorously that f reaches its minimum at the critical point $(1, -\frac{1}{2})$? Consider the disk of radius 2

$$D' = \{(x, y) \mid x^2 + y^2 \leq 4\}$$

which is closed and bounded. On D', f has one critical point $(1, -\frac{1}{2})$. On the boundary of D', which is the circle of radius 2

$$\partial D' = \{(x, y) \mid x^2 + y^2 = 4\}$$

the function f has a lower bound:

$$f(x, y) = x^2 + y^2 + (3 - 2x + y)^2 \geq x^2 + y^2 = 4.$$

Therefore, the minimum of f on D' is $f(1, -\frac{1}{2}) = \frac{3}{2}$. Outside D', that is on the region $x^2 + y^2 > 4$, the same bound works:

$$f(x, y) = x^2 + y^2 + (3 - 2x + y)^2 \geq x^2 + y^2 > 4.$$

This proves that f reaches a global minimum on $\mathbb{R}^2$ at the point $(1, -\frac{1}{2})$, with value $f(1, -\frac{1}{2}) = \frac{3}{2}$. $\square$

Remark 3.3. One can at least check that $(1, -\frac{1}{2})$ is a local minimum, using the second derivatives test:

$$f_{xx}(x, y) = 10$$

$$f_{xy}(x, y) = -4$$

$$f_{yy}(x, y) = 4$$

$$\begin{vmatrix} f_{xx}(1, -\frac{1}{2}) & f_{xy}(1, -\frac{1}{2}) \\ f_{xy}(1, -\frac{1}{2}) & f_{yy}(1, -\frac{1}{2}) \end{vmatrix} = \begin{vmatrix} 10 & -4 \\ -4 & 4 \end{vmatrix} = 24 > 0$$

$$f_{xx}(1, -\frac{1}{2}) = 10 > 0.$$

Thus f has a local minimum at $(1, -\frac{1}{2})$.