Calculus 2502A - Advanced Calculus I
Fall 2014
Adams §12.6: The chain rule

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Additional exercises

A1. Let f be the function defined by:

$$f(w,x,y,z) = \left(\sqrt{1+w+4x}, y^3 - wz\right)$$

and g the function defined by:

$$g(x,y) = \left(\log(5x+y), xy^2, x^2 - y\right).$$

Consider the function h defined by:

$$h(w,x,y,z) = g\left(f(w,x,y,z)\right).$$

Find the Jacobian matrix of h at the point $(w,x,y,z) = (4,1,2,1)$.

A2. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function. Assume that f is locally invertible around a point $\vec{a}$, i.e., there is a neighborhood U of $\vec{a}$, a neighborhood V of $f(\vec{a})$, and a function $g\colon V \to U$ which is inverse to the restriction $f|_U \colon U \to V$. Assume moreover that g is differentiable. Show that the Jacobian matrix of g at $f(\vec{a})$ is the inverse of the Jacobian matrix of f at $\vec{a}$:

$$Dg(f(\vec{a})) = \left(Df(\vec{a})\right)^{-1}.$$

A3. Consider the transformation between Cartesian and polar coordinates:

$$\begin{cases} x(r,\theta) = r\cos\theta \\ y(r,\theta) = r\sin\theta. \end{cases}$$

Call this transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, in other words:

$$T(r,\theta) = (x(r,\theta), y(r,\theta)) = (r\cos\theta, r\sin\theta)\,.$$

a) Let us restrict T to $U := \left\{(r,\theta) \in \mathbb{R}^2 \mid r > 0 \text{ and } 0 < \theta < \frac{\pi}{2}\right\}$. On that domain, find an explicit inverse to T

$$T^{-1}\colon V \to U$$

where $V := T(U) = \{(x,y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$ denotes the first quadrant.

b) Compute the Jacobian matrix of T and of T^{-1} at arbitrary points of their respective domains.

c) Verify the conclusion of Exercise A2 for this example. In other words, using part (b), check the equality:

$$D(T^{-1})(r\cos\theta, r\sin\theta) = (DT(r,\theta))^{-1}$$

for every $(r,\theta) \in U$.