Quillen cohomology of Π-algebras

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Definition

Let Π be the full subcategory of the homotopy category of pointed spaces consisting of finite wedges of spheres $\bigvee S^{n_i}$, $n_i \geq 1$. A **Π-algebra** is a product-preserving functor $\Pi^{\text{op}} \to \mathbf{Set}_*$.

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Realization problem: Given a Π-algebra A, is there a space X such that $\pi_* X \simeq A$ as Π-algebras? If so, can we classify them?

Obstruction theory

Blanc-Dwyer-Goerss (2004) provided an obstruction theory to realizing a Π-algebra A, where the obstructions live in Quillen cohomology of A. They build the moduli space of all realizations as a holim of moduli spaces of “potential n-stages”.

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- Moduli space of potential 0-stages $\simeq \mathrm{BAut}(A)$.
- For a potential $(n-1)$-stage Y, there is an obstruction $o_Y \in \mathrm{HQ}^{n+2}(A; \Omega^n A)/\mathrm{Aut}(A, \Omega^n A)$ to lifting Y to a potential n-stage.
- If o_Y vanishes, the lifts of Y are "classified" by $\mathrm{HQ}^{n+1}(A; \Omega^n A)$, i.e. it acts transitively on the set of lifts.

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Goal: Run this obstruction theory in simple cases.

Truncation

Want to understand the obstruction groups better. What does Quillen cohomology of Π-algebras look like?

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Avoiding the problem: Look at truncated Π-algebras only.

Let $\Pi\mathbf{Alg}_1^n$ denote the category of n-truncated Π-algebras. We have the adjunction

$$\Pi\mathbf{Alg} \underset{\iota_n}{\overset{P_n}{\rightleftarrows}} \Pi\mathbf{Alg}_1^n$$

where P_n is the n^{th} Postnikov truncation functor and ι_n is the inclusion.

Truncation

Proposition (F.)

If a module M over a Π-algebra A is n-truncated, then there is a natural isomorphism

$$\mathrm{HQ}^*_{\Pi\mathbf{Alg}_1^n}(P_n A; M) \xrightarrow{\cong} \mathrm{HQ}^*_{\Pi\mathbf{Alg}}(A; M).$$

Proof sketch: The left Quillen functor $P_n : s\Pi\mathbf{Alg} \to s\Pi\mathbf{Alg}_1^n$ preserves <u>all</u> weak equivalences, not just between cofibrant objects. □

Application to 2-types

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Proposition

Weak homotopy types of realizations of A are in bijection with $\mathrm{H}^3(A_1; A_2)/\mathrm{Aut}(A)$.

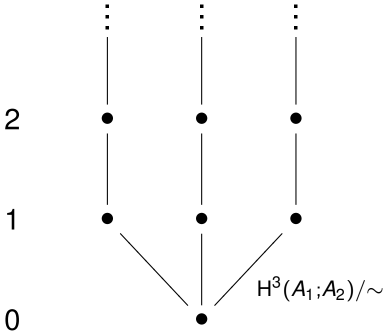
Proof sketch: Potential 0-stage is unique up to homotopy; obstruction to lifting vanishes. Lifts to 1-stages are classified by

$$\mathrm{HQ}^2(A; \Omega A) \cong \mathrm{HQ}^2_{\mathbf{Gp}}(A_1; A_2) \cong \mathrm{H}^3(A_1; A_2).$$

The indeterminacy is the action of $\pi_1 \mathrm{BAut(A)} = \mathrm{Aut}(A)$. Since $\Omega^2 A = 0$, all further obstructions vanish. $\square$

Application to 2-types

Realization tree for A



Application to 2-types

We recover a classic result of MacLane-Whitehead on homotopy 2-types. Our argument generalizes to the following case, for any $n \geq 2$.

Theorem

Let A be a Π-algebra with A_1, A_n and zero elsewhere. Then A is realizable and (weak) homotopy types of realizations are in bijection with $H^{n+1}(A_1; A_n)/\mathrm{Aut}(A)$.

Application to 2-types

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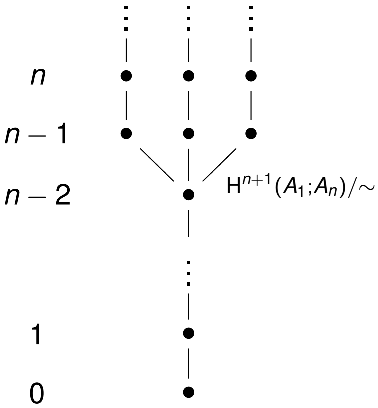

Application to 2-types

The uniqueness obstructions can be identified with k-invariants of the realizations, using work of H.J. Baues and D. Blanc comparing different obstruction theories.

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First goal: Compute HH^* for 2-truncated Π-algebras.

Extended group cohomology

A module $\begin{pmatrix} M_2 \\ M_1 \end{pmatrix}$ over $\begin{pmatrix} A_2 \\ A_1 \end{pmatrix}$ is the data of A_1-modules M_1 and M_2 and an action map $A_2 \otimes M_1 \to M_2$ which is A_1-equivariant.

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For group cohomology, the short exact sequence of G-modules

$$0 \to I_G \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

yields $\mathrm{HH}^i(G; M) \cong \mathrm{H}^{i+1}(G; M) = \mathrm{Ext}^{i+1}(\mathbb{Z}, M)$ for $i \geq 1$.

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Similarly in the 2-truncated case, there is a "constant module" $\begin{pmatrix} 0 \\ \mathbb{Z} \end{pmatrix}$ and for $i \geq 1$ an isomorphism

$$\mathrm{HH}^i \left(\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} ; \begin{pmatrix} M_2 \\ M_1 \end{pmatrix} \right) \cong \mathrm{H}^{i+1} \left(\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} ; \begin{pmatrix} M_2 \\ M_1 \end{pmatrix} \right) .$$

Extended group cohomology

$$\mathrm{H}^* \left(\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} ; \begin{pmatrix} M_2 \\ M_1 \end{pmatrix} \right) := \mathrm{Ext}^* \left(\begin{pmatrix} 0 \\ \mathbb{Z} \end{pmatrix} , \begin{pmatrix} M_2 \\ M_1 \end{pmatrix} \right)$$

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We can use a bar resolution of $\begin{pmatrix} 0 \\ \mathbb{Z} \end{pmatrix}$ to relate the computations to familiar homological algebra.

Moral

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- The obstruction theory of Blanc-Dwyer-Goerss provides a fresh perspective on a classic problem, and a useful theoretical tool.
- Brute force computations for Quillen cohomology of Π-algebras can be unwieldy.

Work in progress

- Compute extended group cohomology and HQ^* of 2-truncated Π-algebras.
- Study the case of an arbitrary 2-stage Π-algebra.
- Relate the Quillen cohomology groups involved to cohomology of Eilenberg-MacLane spaces.
- Study some 3-stage examples.

Further questions

- Existence obstructions
- Algebraic models
- Rational case
- Stable analogue
- Operations in Quillen cohomology

Thank you!

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