The mod p motivic Steenrod algebra in characteristic p

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Outline

Note: Work in progress.

1. Setup
2. Hopkins–Morel isomorphism
3. Motivic dual Steenrod algebra
4. Reduction step
5. Sketch of ideas

Setup

Setup: motivic homotopy theory.

Work over a nice scheme S, called the *base scheme*. (Noetherian, separated, and of finite Krull dimension.)

Sm_S = category of smooth schemes of finite type over S.

Motivic spaces = localization of simplicial presheaves on Sm_S at $\mathbb{A}^1$-equivalences and Nisnevich hypercovers.

Motivic spectra = stabilization of pointed motivic spaces with respect to $S^1 \wedge -$ and $\mathbb{G}_m \wedge -$.

Let $\mathrm{SH}(S)$ denote the motivic stable homotopy category over S. It is a compactly generated tensor triangulated category.

Bigraded spheres

Motivic spheres:

$$S^{p,q} = (S^1)^{\wedge(p-q)} \wedge_S (\mathbb{G}_m)^{\wedge q}$$

and corresponding suspension functors:

$$\Sigma^{p,q} X = S^{p,q} \wedge_S X = X(q)[p].$$

Unit for the smash product $\wedge_S$:

$$\mathbb{1}_S = S^{0,0} = \Sigma^\infty S_+.$$

Example.

- $\mathbb{G}_m = \mathbb{A}^1 - \{0\} = S^{1,1}$.
- $\mathbb{A}^n - \{0\} \simeq S^{2n-1,n}$.
- $\mathbb{P}^1 \simeq \mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \simeq S^1 \wedge \mathbb{G}_m = S^{2,1}$.

Motivic Eilenberg-MacLane spectra

$H\mathbb{Z}$ is a motivic spectrum representing motivic cohomology in $\mathrm{SH}(S)$. $H\mathbb{Z}$ is an E_∞ ring spectrum, in an essentially unique way. Likewise for $H\mathbb{F}_p$.

This yields well-behaved categories of modules over $H\mathbb{Z}$ or $H\mathbb{F}_p$.

Notation: In case of ambiguity, write the dependency on the base scheme as $H\mathbb{F}_p^S$.

Remark. $H\mathbb{F}_l$ has complicated homotopy:

$$\begin{aligned}\pi_{p,q}H\mathbb{F}_l &= \mathrm{SH}(S)\left(S^{p,q}, H\mathbb{F}_l\right)\\ &= \mathrm{SH}(S)\left(S^{0,0}, \Sigma^{-p,-q}H\mathbb{F}_l\right)\\ &= H^{-p,-q}(S;\mathbb{F}_l),\end{aligned}$$

motivic cohomology of the base scheme S.

Hopkins–Morel isomorphism

Motivation: the Hopkins–Morel isomorphism.

In classical homotopy theory: complex cobordism MU, with

$$\pi_* MU \cong \mathbb{Z}[a_1, a_2, \ldots], \ |a_i| = 2i.$$

The orientation map $MU \to H\mathbb{Z}$ induces a map

$$MU/(a_1, a_2, \ldots) \xrightarrow{\simeq} H\mathbb{Z}$$

which is an equivalence.

Hopkins–Morel, II

In motivic homotopy theory: algebraic cobordism spectrum MGL, with $a_i \in \pi_{2i,i}\mathrm{MGL}$.

The orientation map $\mathrm{MGL} \to H\mathbb{Z}$ induces a map

$$\Phi\colon \mathrm{MGL}/(a_1, a_2, \ldots) \to H\mathbb{Z}$$

Theorem (Hopkins–Morel 2004; Hoyois 2015)**.** Let S be essentially smooth over a field $\Bbbk$.

1. In the case $\mathrm{char}(\Bbbk) = 0$, then Φ is an equivalence.
2. In the case $\mathrm{char}(\Bbbk) = p$, then Φ becomes an equivalence after inverting p.

Remark. This theorem has interesting applications to the slice filtration.

Hopkins–Morel, III

Key step: For any prime number $l \neq p$,

$$H\mathbb{F}_l \wedge \Phi \colon H\mathbb{F}_l \wedge \mathrm{MGL}/(a_1, a_2, \ldots) \to H\mathbb{F}_l \wedge H\mathbb{Z}$$

is an equivalence.

Key ingredient: Motivic Steenrod algebra and its dual.

Motivic dual Steenrod algebra

There are certain classes in $\pi_{*,*}(H\mathbb{F}_l \wedge H\mathbb{F}_l)$

$$\tau_i, \text{ with } |\tau_i| = (2l^i - 1, l^i - 1), i \geq 0$$
$$\xi_i, \text{ with } |\xi_i| = (2l^i - 2, l^i - 1), i \geq 1.$$

Consider sequences $I = (\epsilon_0, r_1, \epsilon_1, r_2, \epsilon_2, \ldots)$ with $\epsilon_i \in \{0,1\}$, $r_i \geq 0$, and only finitely many non-zero terms. Consider monomials of the form

$$\tau_0^{\epsilon_0}\xi_1^{r_1}\tau_1^{\epsilon_1}\xi_2^{r_2}\tau_2^{\epsilon_2}\cdots \in \pi_{*,*}(H\mathbb{F}_l \wedge H\mathbb{F}_l).$$

Consider the induced map of $H\mathbb{F}_l$-modules

$$\bigoplus_{\text{such sequences } I} \Sigma^{p_I,q_I} H\mathbb{F}_l \xrightarrow{\psi^S} H\mathbb{F}_l \wedge H\mathbb{F}_l.$$

Note that the indexing set is the same for any base scheme S.

Motivic dual Steenrod algebra, II

Theorem (Voevodsky 2003; Hoyois–Kelly–Østvær 2013)**.** Assume l is invertible on the base scheme S. Then the map ψ^S is an equivalence of $H\mathbb{F}_l$-modules.

Goal. Show that ψ^S is an equivalence in the case $S = \mathrm{Spec}(\Bbbk)$ where $\Bbbk$ is a field of characteristic p, and $l = p$.

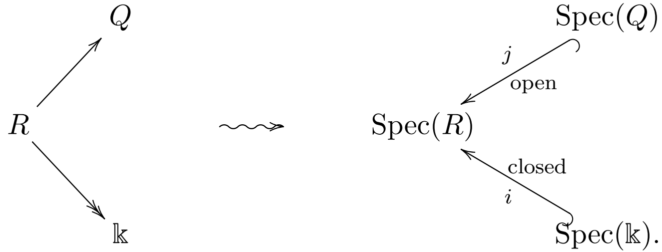
Strategy

Let $\Bbbk$ be a field of characteristic p.

Let R be a discrete valuation ring R having $\Bbbk$ as residue field, and a fraction field $Q = \mathrm{Frac}(R)$ of characteristic zero.

Example. $\Bbbk = \mathbb{F}_p$, $R = \mathbb{Z}_p$, the p-adic integers, and $Q = \mathbb{Q}_p$, the p-adic rationals.

Consider the ring maps and induced maps of affine schemes:



The ingredients

What happens on the generic point Spec(Q)?

Lemma. There is an equivalence

$$j^* H\mathbb{F}_p^R \simeq H\mathbb{F}_p^Q.$$

What happens on the closed point Spec($\Bbbk$)?

Proposition (Spitzweck 2013)**.**

1. There is an equivalence

$$i^* H\mathbb{F}_p^R \simeq H\mathbb{F}_p^{\Bbbk}.$$

2. There is an equivalence of $H\mathbb{F}_p^{\Bbbk}$-module spectra

$$i^! H\mathbb{F}_p^R \simeq \Sigma^{-2,-1} H\mathbb{F}_p^{\Bbbk}.$$

Proposition (F.–Spitzweck)**.** There is an equivalence of $H\mathbb{F}_p^{\Bbbk}$-module spectra

$$i^* j_* H\mathbb{F}_p^Q \simeq H\mathbb{F}_p^{\Bbbk} \oplus \Sigma^{-1,-1} H\mathbb{F}_p^{\Bbbk}.$$

Reduction step

Definition. An object of $\mathrm{SH}(S)$ is smooth if it lies in the full localizing triangulated subcategory of $\mathrm{SH}(S)$ generated by the strongly dualizable objects.

Remark. In the classical stable homotopy category, *every* object is smooth in this sense.

In fact, this is one of the axioms of a stable homotopy theory in the sense of Hovey–Palmieri–Strickland.

If $\Bbbk$ is a field of characteristic zero, then every object of $\mathrm{SH}(\Bbbk)$ is smooth (Röndigs–Østvær 2008).

Reduction step, II

Proposition (F.–Spitzweck). **If** $H\mathbb{F}_p^R$ is smooth in $\mathrm{SH}(R)$, then the map of $H\mathbb{F}_p^{\Bbbk}$-module spectra

$$\bigoplus_I \Sigma^{p_I,q_I} H\mathbb{F}_p^{\Bbbk} \xrightarrow{\psi^{\Bbbk}} H\mathbb{F}_p^{\Bbbk} \wedge H\mathbb{F}_p^{\Bbbk}$$

is an equivalence. In other words, the structure theorem for the dual Steenrod algebra holds for $S = \mathrm{Spec}(\Bbbk)$.

Our goal would be achieved!

Proof idea: Use the previous ingredients and the six functor formalism.

Why smoothness?

Lemma. Let $f\colon S \to T$ be a map of schemes. For Y in $\mathrm{SH}(T)$ and X in $\mathrm{SH}(S)$, consider the natural map in $\mathrm{SH}(T)$

$$\alpha\colon (f_*X) \wedge_T Y \to f_*\left(X \wedge_S f^*Y\right).$$

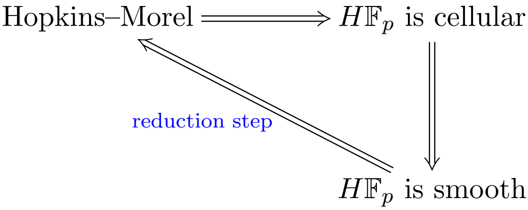
If Y is a smooth object of $\mathrm{SH}(T)$, then α is an isomorphism.

In other words, Y and f_* satisfy the *projection formula*

$$(f_*X) \wedge_T Y \cong f_*\left(X \wedge_S f^*Y\right).$$

More on smoothness

Remark. The Hopkins–Morel isomorphism implies that $H\mathbb{Z}$ and $H\mathbb{F}_p$ are *cellular*, a condition which implies smoothness.



Sketch of ideas

How to prove the new goal, i.e., that $H\mathbb{F}_p^R$ is smooth in $\mathrm{SH}(R)$?

$$H\mathbb{Z} = \mathrm{hocolim}_n\, \Sigma^{-2n,-n}\Sigma^\infty K(\mathbb{Z}, 2n, n)$$

Note: The motivic Eilenberg-MacLane space $K(\mathbb{Z}, p, q)$ is also known as $K(\mathbb{Z}(q), p)$.

$\Rightarrow$ It suffices to show that $\Sigma^\infty K(\mathbb{Z}, 2n, n)$ is smooth for n large enough.

Motivic Dold–Thom

In classical homotopy theory, the Dold–Thom theorem yields:

$$\mathrm{Sym}^\infty(S^n) \simeq K(\mathbb{Z}, n).$$

In motivic homotopy theory, there is an analogue of the Dold–Thom theorem (Suslin–Voevodsky 1996).

In characteristic zero, we have:

$$\mathrm{Sym}^\infty\left((\mathbb{P}^1)^{\wedge n}\right) \simeq \mathrm{Sym}^\infty(S^{2n,n}) \simeq K(\mathbb{Z}, 2n, n).$$

In characteristic $p > 0$, there is an analogous (but more complicated) construction involving correspondences.

$$\mathrm{Sym}^\infty(X) = \mathrm{hocolim}_k \, \mathrm{Sym}^k(X)$$

$\Rightarrow$ It suffices to show that an appropriate analogue of $\Sigma^\infty \, \mathrm{Sym}^k\left((\mathbb{P}^1)^{\wedge n}\right)$ is smooth for n and k large enough.

Resolutions of symmetric powers

Plan of attack:

- Given X in Sm_S, encode k-fold iterated blow-ups in X via an algebraic space B_k.
- Blow up B_k several times, along a stratification similar to that of the fat diagonal in the product X^k. Obtain a nicer algebraic space $\tilde{B}_k$.
- Find a good approximation $\tilde{B}_k \to Z$ by a smooth scheme Z, which is projective if X is.
- Build an appropriate iterated homotopy pushout P_k of smooth projective schemes, which implies that $\Sigma^\infty P_k$ is smooth in $\mathrm{SH}(R)$.
- In the case $X = (\mathbb{P}^1)^{\wedge n}$, show that P_k is a good approximation of $\mathrm{Sym}^k(X)$.

Thank you!