Fibered category of Beck modules*

Martin Frankland
Department of Mathematics
Massachusetts Institute of Technology
franklan@math.mit.edu

April 11, 2010

Beck modules provide a convenient notion of coefficient module to be used in cohomology theories. The notion makes sense in a broad context, and recovers the usual notions of module in familiar settings, such as groups, rings, commutative rings, Lie algebras, etc. Instead of considering the category of modules over an object, one can fit them together into a category of modules over all objects. In these notes, we study that construction and its properties. The notes are meant as an informal document, which includes many questions and comments while omitting many proofs. Their main purpose is to present some ideas and promote discussion.

First, some personal motivation. I've been working with Π-algebras, which are graded groups with additional structure similar to that of the homotopy groups of a space. I'm particularly interested in truncated Π-algebras, those whose groups are zero above some dimension n. For example, the data of a 2-truncated Π-algebra is precisely a group π_1 and a module π_2 over it. Morphisms are pairs of maps $(f_1 : \pi_1 \to \pi'_1, f_2 : \pi_2 \to \pi'_2)$ such that f_2 is f_1-equivariant, i.e. $f_2(g \cdot a) = f_1(g) \cdot f_2(a)$. The same data also describes a Π-algebra concentrated in dimensions 1 and n, for any $n > 1$. Thus that category, whose objects consist of a group and a module over it, appears in the study of certain homotopy types. Baues denotes it $\mathbf{Mod}_{\mathbb{Z}}^{\wedge}$ [2, def I.1.7]; we denote it **ModGp**, the category of modules over groups. I wanted to generalize that construction to **Mod**C, the category of modules over objects of C, and study its properties.

*Notes for a talk at the Graduate Student Topology and Geometry Conference 2010, University of Michigan, Ann Arbor.

Section 1 gives background material on Beck modules and some related constructions. Section 2 gives examples in familiar categories. Section 3 presents the main construction – the fibered category of Beck modules – and some of its properties. Section 4 investigates limits in that fibered category. Section 5 is a digression about a representability property for Beck modules. Section 6 explains how the construction looks vaguely like the tangent bundle of a manifold. Section 7 reformulates a familiar construction of Quillen's in terms of our construction.

1 Beck modules

A good reference about Beck modules and what we can do with them is [1, chap 6], especially section 6.1 therein.

We work in a category C with finite limits.

Definition 1.1. A **Beck module** over an object X of C is an abelian group object in the slice category C/X.

Notation. When it's clear which category C we're working in, the category of Beck modules over X will sometimes be denoted $\mathbf{Mod}_X$ instead of $Ab(C/X)$.

1.1 Functoriality

Proposition 1.2. *If $F : C \to \mathcal{D}$ is a pullback-preserving functor , then for any object X in C, it induces a functor on modules*

$$F_X : Ab(C/X) \to Ab(\mathcal{D}/F(X)).$$

Moreover, F_X is additive.

Proof. F induces a functor $C/X \to \mathcal{D}/F(X)$, which automatically preserves the terminal object (namely, the identity). This functor preserves finite products iff F preserves pullbacks over X, hence it suffices to prove the "absolute" version, i.e. if $F : C \to \mathcal{D}$ preserves finite products (including the terminal object), then it induces an additive functor

$$F : Ab(C) \to Ab(\mathcal{D}).$$

If A is an abelian group object in C, then F applied to the structure maps of A yields structure maps for $F(A)$, and F applied to the structure (condition) diagrams of A yields structure diagrams for $F(A)$.

Addition in $\mathrm{Hom}_{Ab(\mathcal{D})}(FM, FN)$ is given by $(\mu_{FN})_* = (F(\mu_N))_*$, hence F is additive. In other words, for $f, g \in \mathrm{Hom}_{Ab(C)}(M, N)$, we have

$$\begin{aligned} F(f+g) &= F(\mu_N(f \times g)\Delta_N) \\ &= F(\mu_N)F(f \times g)F(\Delta_N) \\ &= \mu_{FN}(Ff \times Fg)\Delta_{FN} \\ &= Ff + Fg. \end{aligned}$$

□

Remark 1.3. By abuse of notation, we called the induced functor F also. The notation $Ab(F)$ or F_* might have been more appropriate, although more cumbersome.

In fact, the condition of preserving pullbacks (including finite products and the terminal object) is too strong. Recall that equalizers are a special case of pullbacks. If $f, g : X \to Y$ are two maps, their equalizer is the pullback

$$\begin{tikzcd} Eq(f,g) \arrow[r] \arrow[d] & X \arrow[d, "(\mathrm{id},g)"] \\ X \arrow[r, "(\mathrm{id},f)"'] & X \times Y \end{tikzcd}$$

which can be thought of as the "intersection of the graphs of f and g". Thus preserving pullbacks is the same as preserving finite limits (assuming there is a terminal object). In particular, the induced functor is then left exact.

In order for F to induce a functor $F : Ab(C/X) \to Ab(\mathcal{D}/FX)$ on Beck modules, it suffices that F preserve kernel pairs of split epimorphisms. By **kernel pair** of a map, we mean the pullback of the map along itself.

Non-example 1.4. Consider the functor $F : \mathbf{Gp} \to \mathbf{Gp}$ that associates to a group the free group on its underlying set, i.e. the comonad of the Free/Forget adjunction. Then F does NOT induce a functor on Beck modules.

To see this, use the fact (2.4) that a Beck module over a group G is a split extension $p : E \to G$ (plus the data of the splitting) with abelian kernel, which can be viewed as the semi-direct product $G \ltimes M \to G$. Take the trivial group 1 and the module $A \to 1$ over it, where A is any abelian group. Apply F to it. The resulting split extension

$$F(A) \to F(1)$$

does NOT have an abelian kernel. Indeed, $F(A)$ is the group of words on elements of A and their formal inverses, and the kernel of the map is the subgroup of words whose exponents add to zero, e.g. $a^2b^{-3}c$. That subgroup is highly non-abelian (as long as A is non-trivial), e.g. the elements ab^{-1} and $a^{-1}b$ do not commute.

1.2 Changing the ground object

We want to compare modules over different objects, using maps between them.

Proposition 1.5. *If C has all pullbacks, then any map $f : X \to Y$ in C induces a pullback functor on Beck modules*

$$f^* : Ab(C/Y) \to Ab(C/X).$$

Moreover, f^* is additive.

Proof. First note that f induces a pullback functor on the slice categories

$$f^* : C/Y \to C/X$$

which is right adjoint to the so-called direct image functor

$$f_! : C/X \to C/Y$$

given by postcomposition. Indeed, consider the following commutative diagram:

$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[dr, "g"'] & f^*Z = X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "p"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

We have the following correspondence:

$$\begin{aligned}
&\mathrm{Hom}_{C/X}\left(W \xrightarrow{g} X, f^*(Z \xrightarrow{p} Y)\right) \\
&\quad = \mathrm{Hom}_{C/X}(W \xrightarrow{g} X, f^*Z \xrightarrow{\pi_1} X) \\
&\quad = \{\varphi : W \to f^*Z \mid g = \pi_1\varphi\} \\
&\quad = \{\varphi_1 = \pi_1\varphi : W \to X, \varphi_2 = \pi_2\varphi : W \to Z \mid f\varphi_1 = p\varphi_2, \varphi_1 = g\} \\
&\quad = \{\varphi_2 : W \to Z \mid fg = p\varphi_2\} \\
&\quad = \mathrm{Hom}_{C/Y}(W \xrightarrow{fg} Y, Z \xrightarrow{p} Y) \\
&\quad = \mathrm{Hom}_{C/Y}\left(f_!(W \xrightarrow{g} X), Z \xrightarrow{p} Y\right).
\end{aligned}$$

As we've seen in the proof of 1.2, any limit-preserving functor induces an additive functor on the categories of abelian group objects, hence the conclusion. $\square$

Definition 1.6. If f^* has a left adjoint, we call it the **pushforward**

$$f_* : Ab(C/X) \to Ab(C/Y).$$

In most interesting cases, C has all pushforwards. It's something we want to assume in our setup.

1.3 Abelianization

Definition 1.7. We call **abelianization** functor Ab_X the left adjoint of the forgetful functor $U_X : Ab(C/X) \to C/X$, if it exists.

In our setup, we assume C has all abelianizations, i.e. Ab_X exists for all object X. Assume moreover that $Ab(C/X)$ is an abelian category for all object X. That holds for example when C is exact [1, chap 2, thm 2.4].

Proposition 1.8. *Let $f : X \to Y$ be a morphism in C and assume f has a pushforward f_*. Then we have:*

$$Ab_Y(X \xrightarrow{f} Y) = f_* Ab_X(X \xrightarrow{\mathrm{id}} X).$$

More generally, the following diagram commutes.

$$\begin{tikzcd} C/X \arrow[r, "f_!"] \arrow[d, "Ab_X"'] & C/Y \arrow[d, "Ab_Y"] \\ Ab(C/X) \arrow[r, "f_*"'] & Ab(C/Y). \end{tikzcd}$$

Proof. All the functors in this diagram are left adjoints of the following functors:

$$\begin{tikzcd} C/X & C/Y \arrow[l, "f^*"] \\ Ab(C/X) \arrow[u, "U_X"] & Ab(C/Y) \arrow[l, "f^*"'] \arrow[u, "U_Y"'] \end{tikzcd}$$

This diagram commutes on the nose, by construction. Since adjoint pairs compose, and adjoints are unique up to unique iso, we conclude that the diagram of left adjoints also commutes. (Minor quibble: It only commutes up to natural iso, but the abelianization and pushforward functors can be chosen so that it commutes on the nose). $\square$

2 Examples

In this section, we describe Beck modules, abelianization functors, and pushforwards in various categories. Again, [1, chap 6] is a good reference.

2.1 Sets

As a toy example, let's look at the category **Set** of sets.

Proposition 2.1. *A Beck module over a set X is a "bundle of abelian groups over X", i.e. a map of sets $E \to X$ where each fiber E_x has a structure of abelian group.*

Proof. We have a map of sets $p : E \to X$, with its structure maps. The unit map is a section $e : X \to E$ of p. The fiber product:

$$\begin{tikzcd} E \times_X E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & X \end{tikzcd}$$

is just the set of pairs of points E lying in the same fiber, i.e.

$$E \times_X E = \{(e, e') \mid p(e) = p(e')\}.$$

Hence the structure maps describe precisely the structure of an abelian group on each fiber E_x of p, the unit e being the zero section. □

In other words, a Beck module over the set X is an X-indexed family of abelian groups E_x, or a functor from the discrete category X into **Ab**.

Proposition 2.2. *The abelianization functor Ab_X : $\mathbf{Set}/X \to Ab(\mathbf{Set}/X)$ is the fiberwise free abelian group functor, i.e. for $Z \to X$, the abelian group $(Ab_X Z)_x$ over a point $x \in X$ is the free abelian group on the fiber Z_x.*

Proof. Let $Z \to X$ be in $\mathbf{Set}/X$ and $E \to X$ a module over X, then the following holds:

$$
\begin{aligned}
&\mathrm{Hom}_{Ab(\mathbf{Set}/X)}(\mathrm{FibFreeAb}(Z \to X), E \to X)\\
&= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{FibFreeAb}(Z \to X)_x, E_x)\\
&= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{FreeAb}(Z_x), E_x)\\
&= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Set}}(Z_x, U(E_x))\\
&= \mathrm{Hom}_{\mathbf{Set}/X}(Z \to X, U_X(E \to X)).
\end{aligned}
$$

□

Proposition 2.3. *For a map of sets $f : X \to Y$, the pushforward functor f_* is as follows. If $E \to X$ is a module over X, then the module f_*E over Y is given by:*

$$
(f_*E)_y = \bigoplus_{x \in f^{-1}(y)} E_x
$$

Proof. Let $M \to Y$ be a module over Y, let E' denote the module defined by the above formula.

$$
\begin{aligned}
\mathrm{Hom}_{Ab(\mathbf{Set}/Y)}(E', M) &= \prod_{y \in Y} \mathrm{Hom}_{\mathbf{Ab}}(E'_y, M_y)\\
&= \prod_{y \in Y} \prod_{x \in f^{-1}(y)} \mathrm{Hom}_{\mathbf{Ab}}(E_x, M_y)\\
&= \prod_{x \in X} \mathrm{Hom}_{\mathbf{Ab}}(E_x, M_{f(x)})\\
&= \mathrm{Hom}_{Ab(\mathbf{Set}/X)}(E, f^*M).
\end{aligned}
$$

□

2.2 Groups

There is a major difference between the example of sets and groups. In groups (and any structure having underlying groups), the addition structure map of a Beck module is determined by the underlying groups, and hence provides no additional data.

Proposition 2.4. *A Beck module over a group G is a split extension of G (with the data of a splitting) with abelian kernel:*

$$1 \longrightarrow K \longrightarrow E \underset{e}{\overset{p}{\rightleftarrows}} G \longrightarrow 1.$$

This category is equivalent to the standard category of (say, left) modules over G. To a split extension, one associates the kernel K with induced action from G, given by $g \cdot k = e(g)k$. To a usual module K, one associates the semidirect product $G \ltimes K \to G$. Note that this is the same as a module over the group ring $\mathbb{Z}G$. It's also the same as (covariant) functors from the one-object category G to Ab.

Proposition 2.5. *For a map of groups $f : H \to G$, the pushforward functor f_* associates to an H-module M the G-module:*

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M.$$

Proposition 2.6.
1. *The module $Ab_X X$ of a group G is the augmentation ideal $I_G = \ker(\mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z})$.*
2. *The abelianization functor $Ab_G : \mathbf{Gp}/G \to Ab(\mathbf{Gp}/G)$ associates to $H \to G$ the G-module:*
$$\mathbb{Z}G \otimes_{\mathbb{Z}H} I_H$$

Let us make the correspondence more precise. For groups, "global sections" correspond to crossed homomorphisms:

$$\begin{aligned}\Gamma(G, M) &= \mathrm{Hom}_{\mathbf{Gp}/G}(G \xrightarrow{\mathrm{id}} G, G \ltimes M \to G)\\ &\cong \{\varphi : G \to M \mid \varphi(gg') = \varphi(g) + g \cdot \varphi(g')\}.\end{aligned}$$

Such a crossed homomorphism corresponds, via the adjunction, to the G-module map $\alpha : I_G \to M$ defined by:

$$\alpha(1 - g) = \varphi(g).$$

2.3 Abelian groups

Something interesting happens in the case of abelian groups because they form an abelian category. We'll come back to that in section 6.

Proposition 2.7. *A Beck module over an abelian group A is a split extension of A in abelian groups:*

$$0 \longrightarrow K \longrightarrow E \cong A \oplus K \underset{e}{\overset{p}{\rightleftarrows}} A \longrightarrow 0.$$

The proof is basically the same as for groups, except the "total space" has to be abelian, which forces the extension to be trivial.

This category is equivalent to **Ab**. To a split extension, one associates the kernel K. To an abelian group K, one associates the projection $A \oplus K \to A$. It's also the same as functors from the trivial category $*$ to Ab.

Proposition 2.8. *For a map of abelian groups $f : A \to B$, the pushforward functor f_* is the identify functor on Ab, i.e. sends $A \oplus K$ to $B \oplus K$.*

Proof. The pullback functor f^* is the identity on **Ab**, under the above identification. □

Proposition 2.9. *The abelianization functor $Ab_A : \mathbf{Ab}/A \to Ab(\mathbf{Ab}/A) \cong \mathbf{Ab}$ is the "source" functor, which sends $B \to A$ to B.*

Proof. Under the above identification, the forgetful functor $U_A : \mathbf{Ab} \to \mathbf{Ab}/A$ sends an abelian group K to $A \oplus K \xrightarrow{\pi_1} A$. Thus we have:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Ab}/A}(B \to A, U_A(K)) &= \mathrm{Hom}_{\mathbf{Ab}/A}(B \to A, A \oplus K \to A) \\
&= \mathrm{Hom}_{\mathbf{Ab}}(B, K).
\end{aligned}$$

□

2.4 Associative algebras

By **ring**, we will mean by default an associative, unital ring. Fix a ground ring R which is commutative, and consider the usual notion of **associative R-algebra**, i.e. a ring A which is also an R-module in a compatible way. Equivalently, it is a ring A with a ring map $R \to A$ which lands in the center of A.

Notation. Let $\mathbf{Alg}_R$ denote the category of R-algebras.

Proposition 2.10. *A Beck module over an associative R-algebra A is a split extension of A (with the data of a splitting) with square zero kernel:*

$$0 \longrightarrow M \longrightarrow E \cong A \oplus M \underset{s}{\overset{p}{\rightleftarrows}} A \longrightarrow 0.$$

Equivalently, it is the data of an A-bimodule M over R, i.e. the two actions coincide for scalars (elements of R).

Notation. For an R-algebra A, denote by $m : A \otimes_R A \to A$ the multiplication map, and let $I_A := \ker(A \otimes_R A \xrightarrow{m} A)$ be its kernel.

Proposition 2.11. *Abelianization in R-algebras is given by $Ab_A A = I_A$.*

Let us describe the correspondence more precisely. First, "global sections" $\mathrm{Hom}_{\mathbf{Alg}_R/A}(A \xrightarrow{\mathrm{id}} A, A \oplus M \xrightarrow{p} A)$ are just R-linear derivations $\mathrm{Der}_R(A, M)$. There is a natural equivalence of R-modules

$$\begin{aligned} \alpha : \mathrm{Hom}_{A-\mathbf{Bimod}_R}(I_A, M) &\cong \mathrm{Der}_R(A, M) : \beta \\ f &\mapsto \varphi(a) = f(1 \otimes a - a \otimes 1) \\ f\left(\sum a_i \otimes b_i\right) = \sum a_i \varphi(b_i) &\leftarrow\!\shortmid \varphi \end{aligned}$$

A more elegant way of saying this is that there is a universal R-derivation

$$\begin{aligned} d : A &\to I_A \\ a &\mapsto 1 \otimes a - a \otimes 1 \end{aligned}$$

such that the natural map

$$\begin{aligned} \mathrm{Hom}_{A-\mathbf{Bimod}_R}(I_A, M) &\to \mathrm{Der}_R(A, M) \\ f &\mapsto f \circ d \end{aligned}$$

is an iso, which we called α above.

Proposition 2.12. *The pushforward functor $f_* : A - \mathbf{Bimod}_R \to B - \mathbf{Bimod}_R$ is given by*

$$f_*(M) = B \otimes_A M \otimes_A B$$

equipped with B-multiplication on the left and right.

Corollary 2.13. *The abelianization functor is $Ab_B(A \to B) = B \otimes_A I_A \otimes_A B$.*

2.5 Commutative algebras

Notation. Let $\mathbf{Com}_R$ denote the category of commutative R-algebras.

Proposition 2.14. *A Beck module over a commutative R-algebra A is a split extension of A (with the data of a splitting) with square zero kernel:*

$$0 \longrightarrow M \longrightarrow E \cong A \oplus M \underset{s}{\overset{p}{\rightleftarrows}} A \longrightarrow 0.$$

Equivalently, it is the data of an A-module M in the usual sense (i.e. an abelian group with an action of A).

The proof is the same as for associative algebras 2.10, except the total space E must be commutative, which means the two actions agree:

$$a \cdot m = s(a)m = ms(a) = m \cdot a.$$

Proposition 2.15. *In commutative R-algebraa, abelianization is given by:*

$$Ab_A A = I_A / I_A^2.$$

Notation. The module $Ab_A A = I_A / I_A^2$ representing R-derivations is called the **module of differentials** and denoted $\Omega_{A/R}$.

As for associative R-algebras, the "global sections" of a Beck module $A \oplus M \to A$ over the commutative R-algebra A are precisely the R-derivations $\mathrm{Der}_R(A, M)$. Again, there is a universal R-derivation

$$\begin{aligned} d : A &\to \Omega_{A/R} \\ a &\mapsto 1 \otimes a - a \otimes 1 \end{aligned}$$

such that the natural map

$$\begin{aligned} \mathrm{Hom}_{A-\mathbf{Mod}}(\Omega_{A/R}, M) &\to \mathrm{Der}_R(A, M) \\ f &\mapsto f \circ d \end{aligned}$$

is an iso. Compare [5], the setup before proposition 4.27.

Proposition 2.16. *The pushforward functor* $f_* : A - \mathbf{Mod} \to B - \mathbf{Mod}$ *is given by*

$$f_*(M) = B \otimes_A M$$

equipped with B-multiplication on the left.

Corollary 2.17. *The abelianization functor is* $Ab_B(A \to B) = B \otimes_A \Omega_{A/R}$.

3 Fibered category of Beck modules

Instead of looking only at Beck modules over a single object X of C, we want to know what happens when we change the base object X. For this, we assemble all the categories $\mathbf{Mod}_X$ together.

3.1 Construction

Definition 3.1. The (fibered) **category of Beck modules** over C, denoted $\mathbf{Mod}C$ is the category whose objects are pairs (X, E), where X is an object of C and $E \to X$ is a Beck module over X. A morphism from (X, E) to (Y, E') consists of maps $f : X \to Y$ and $\varphi : E \to E'$ in C making the obvious diagram commute:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d] & E' \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and such that the horizontal arrows respect the group structure maps of $E \to X$ and $E' \to Y$.

Consider the forgetful functor $U : ModC \to C$ taking the pair (X, E) to the base object X. Its fiber over an object X (i.e. subcategory of $ModC$ of objects sent to X and morphisms sent to id_X) is exactly the category $\mathbf{Mod}_X$ of Beck modules over X. Now let's make sure that $ModC$ is indeed fibered over C.

Proposition 3.2. *A morphism from (X, E) to (Y, E') as defined above is the same data as a map $f : X \to Y$ in C and a map $\varphi : E \to f^*E'$ in $\mathbf{Mod}_X$.*

Proof. Consider the commutative diagram:

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr] \arrow[dd, "p"'] & & E' \arrow[dd, "p'"] \\ & f^*E' \arrow[ur] \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] & & Y. \end{tikzcd}$$

Viewing it as a diagram in C, the map φ is in $\mathrm{Hom}_{C/Y}(E \stackrel{fp}{\to} Y, E' \stackrel{p'}{\to} Y) = \mathrm{Hom}_{C/X}(E \stackrel{p}{\to} X, f^*E' \stackrel{f^*p'}{\to} X)$. By construction of the pullback f^*E' and its structure maps, this adjoint map φ is actually in the subset $\mathrm{Hom}_{Ab(C/X)}(E \stackrel{p}{\to} X, f^*E' \stackrel{f^*p'}{\to} X)$ iff the original φ respects the structure maps of $E \to X$ and $E' \to Y$. □

Corollary 3.3. *Pullback squares*

$$\begin{tikzcd} f^*E' \arrow[r, "\varphi"] \arrow[d] & E' \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

are Cartesian morphisms in $ModC$. The forgetful functor $U : ModC \to C$ makes $ModC$ into a fibered category over C in the sense of [8, section 3.1.1]. The system of pullbacks makes it into a cleaved category.

3.2 Pseudofunctor point of view

Following [8, section 3.1.2], one can think of the fibered category $\mathbf{Mod}C$ as a (contravariant) pseudofunctor

$$\mathbf{Mod}_{(-)} : C \to \mathbf{AbCat}$$

associating to each object X of C its category of Beck modules $\mathbf{Mod}_X$ and to each map $f : X \to Y$ the pullback functor $f^* : \mathbf{Mod}_Y \to \mathbf{Mod}_X$. Pseudofunctor means that the pullbacks don't necessarily compose on the nose (i.e. $f^*g^* = (gf)^*$), but rather up to a coherent system of natural isos. The fibered category $\mathbf{Mod}C$ is the Grothendieck construction on this (pseudo)functor. Note that the Grothendieck construction is useful in homotopy theory [4, lecture 3] and in the theory of stacks.

3.3 Relationship to the ground category

Proposition 3.4. *The "zero section" functor $Z : C \to ModC$ which sends X to $(X, 0_X)$ is a both a left and right adjoint of U. In particular, U preserves all limits and colimits.*

Proof. It follows from the fact that 0_X is a zero object in the additive category $\mathbf{Mod}_X$ and the pullback of zero is again zero, i.e. $f^*0_Y = 0_X$.

$$\begin{aligned}\mathrm{Hom}_C(X, U(Y, E')) &= \mathrm{Hom}_C(X, Y)\\ &= \mathrm{Hom}_{ModC}((X, 0_X), (Y, E'))\\ &= \mathrm{Hom}_{ModC}(Z(X), (Y, E')).\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}_C(U(X, E), Y) &= \mathrm{Hom}_C(X, Y)\\ &= \mathrm{Hom}_{ModC}((X, E), (Y, 0_Y))\\ &= \mathrm{Hom}_{ModC}((X, E), Z(Y)).\end{aligned}$$

□

In fact, there is a more general relative version of this proposition, where we work over fixed objects. The previous case is over the terminal object.

Proposition 3.5. *Consider the forgetful functor $U : \mathbf{Mod}C/(X, E) \to C/X$.*

1. *The left adjoint of U is the "zero section" functor Z, which sends $Y \xrightarrow{f} X$ to $(Y, 0_Y) \to (X, E)$.*

2. *The right adjoint of U is the "pullback" functor Pull, which sends $Y \xrightarrow{f} X$ to $(Y, f^*E) \to (X, E)$.*

Proof. 1. Similar to the absolute version:

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{Mod}C/(X,E)}\left(Z(Y \xrightarrow{f} X), (X', E') \xrightarrow{(f',\varphi')} (X, E)\right)\\ &= \mathrm{Hom}_{\mathbf{Mod}C/(X,E)}\left((Y, 0_Y) \xrightarrow{(f,0)} (X, E), (X', E') \xrightarrow{(f',\varphi')} (X, E)\right)\\ &= \{(g, \varphi) : (Y, 0_Y) \to (X', E') \mid (f', \varphi') \circ (g, \varphi) = (f, 0)\}\\ &= \{g : Y \to X' \mid f' \circ g = f\} \qquad \text{since } \varphi \text{ must be 0, and 0 works}\\ &= \mathrm{Hom}_{C/X}\left(Y \xrightarrow{f} X, X' \xrightarrow{f'} X\right)\\ &= \mathrm{Hom}_{C/X}\left(Y \xrightarrow{f} X, U\left((X', E') \xrightarrow{(f',\varphi')} (X, E)\right)\right).\end{aligned}$$

2. We have the following:

$$
\begin{aligned}
&\mathrm{Hom}_{\mathbf{Mod}C/(X,E)}\left((X',E') \overset{(f',\varphi')}{\to} (X,E), Pull(Y \overset{f}{\to} X)\right)\\
&= \mathrm{Hom}_{\mathbf{Mod}C/(X,E)}\left((X',E') \overset{(f',\varphi')}{\to} (X,E), (Y,f^*E) \overset{f}{\to} (X,E)\right)\\
&= \{(g,\varphi) : (X',E') \to (Y,f^*E) \mid f \circ (g,\varphi) = (f',\varphi')\}\,.
\end{aligned}
$$

This consists of the data of $g : X' \to Y$ and a map $E' \to g^*(f^*E) \cong (fg)^*E = (f')^*E$ in $\mathbf{Mod}_{X'}$ making the diagram commute. But since

$$
\begin{tikzcd}
f^*E \arrow[r] \arrow[d] & E \arrow[d] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}
$$

is a pullback square, our map φ must "be" (under the usual identification) the map $\varphi' : E' \to (f')^*E$ in $\mathbf{Mod}_{X'}$. Hence it provides no additional data and no constraint, and the above set of morphisms is:

$$
\begin{aligned}
&\{g : X' \to Y \mid f \circ g = f'\}\\
&= \mathrm{Hom}_{C/X}\left(X' \overset{f'}{\to} X, Y \overset{f}{\to} X\right)\\
&= \mathrm{Hom}_{C/X}\left(U\left((X',E') \overset{(f',\varphi')}{\to} (X,E)\right), Y \overset{f}{\to} X\right).
\end{aligned}
$$

□

We can use this proposition to study the abelianization in $\mathbf{Mod}C$. Let's think of the forgetful functor $U : \mathbf{Mod}C \to C$ sending (X,E) to X as taking the "ground level" part of the data. With this in mind, we'll show that the ground level part of the abelianization is the abelianization of the ground level part.

Lemma 3.6. *Assume $L : C \to \mathcal{D}$ preserves finite products and has a right adjoint $R : \mathcal{D} \to C$. Then the induced functors $L : Ab(C) \to Ab(\mathcal{D})$ and $R : Ab(\mathcal{D}) \to Ab(C)$ still form an adjoint pair.*

Proof. $\mathrm{Hom}_{Ab(C)}(\widetilde{c}, \widetilde{Rd})$ is the subset of $\mathrm{Hom}_C(c,Rd) \cong \mathrm{Hom}_{\mathcal{D}}(Lc,d)$ consisting of maps $c \to Rd$ which commute with the structure maps. So we need to show that this holds iff the adjoint map $Lc \to d$ commutes with structure maps. This is true by the naturality of the adjunction, and the fact that $\widetilde{Lc}$ and $\widetilde{Rd}$ have structure maps induced by those of $\widetilde{c}$ and $\widetilde{d}$, respectively. For example, the diagram of multiplication maps:

$$\begin{tikzcd} c \times c \arrow[r, "\mu_c"] \arrow[d] & c \arrow[d] \\ R(d \times d) \cong Rd \times Rd \arrow[r, "R\mu_d"'] & Rd \end{tikzcd}$$

commutes iff the adjoint diagram:

$$\begin{tikzcd} L(c \times c) \cong Lc \times Lc \arrow[r, "L\mu_c"] \arrow[d] & Lc \arrow[d] \\ d \times d \arrow[r, "\mu_d"'] & d \end{tikzcd}$$

commutes. □

Proposition 3.7. *The following diagram commutes:*

$$\begin{tikzcd} \mathbf{Mod}C/(X,E) \arrow[r, "Ab_{(X,E)}"] \arrow[d, "U"'] & Ab(\mathbf{Mod}C/(X,E)) \arrow[d, "U"'] \\ C/X \arrow[r, "Ab_X"] & Ab(C/X). \end{tikzcd}$$

Proof. By 3.5 and 3.6, the diagram above consists of left adjoints. Let us write all four adjunctions:

$$\begin{tikzcd} \mathbf{Mod}C/(X,E) \arrow[r, shift left, "Ab_{(X,E)}"] \arrow[d, shift right, "U"'] & Ab(\mathbf{Mod}C/(X,E)) \arrow[l, shift left, "U_{(X,E)}"] \arrow[d, shift right, "U"'] \\ C/X \arrow[r, shift left, "Ab_X"] \arrow[u, shift right, "Pull"'] & Ab(C/X). \arrow[l, shift left, "U_X"] \arrow[u, shift right, "Pull"'] \end{tikzcd}$$

The diagram of right adjoints commutes on the nose:

$$U_{(X,E)} \circ Pull = Pull \circ U_X$$

by definition of induced functor on category of abelian group objects. Thus their left adjoints are naturally isomorphic:

$$U \circ Ab_{(X,E)} \cong Ab_X \circ U.$$

As in proposition 1.8, the abelianizations can be chosen so that this is an equality on the nose. □

Proposition 3.8. *The "total space" functor $Src : \mathbf{Mod}C \to C$ which sends $E \xrightarrow{p} Y$ to E (viewed as object in C) has a left adjoint, which sends an object X of C to $Ab_X X \to X$. The notation Src stands for "source".*

Proof. A map in $\mathrm{Hom}_{\mathbf{Mod}C}\left(Ab_X X \to X, E \xrightarrow{p} Y\right)$ consists of a map $f : X \to Y$ and a map $Ab_X X \to f^* E$ in $\mathbf{Mod}_X$.

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Mod}_X}(Ab_X X \to X, f^*E \to X) &= \mathrm{Hom}_{C/X}(X \xrightarrow{\mathrm{id}} X, f^*E \to X) \\
&= \mathrm{Hom}_{C/Y}(f_!(X \xrightarrow{\mathrm{id}} X), E \to Y) \\
&= \mathrm{Hom}_{C/Y}(X \xrightarrow{f} Y, E \to Y)
\end{aligned}$$

Therefore, a map in $\mathrm{Hom}_{\mathbf{Mod}C}\left(Ab_X X \to X, E \xrightarrow{p} Y\right)$ consists of the data of f and φ in such a diagram:

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\
& Y &
\end{tikzcd}$$

which is the data of φ only, since f must be $p\varphi$. Since there is no constraint on φ, we conclude:

$$\mathrm{Hom}_{\mathbf{Mod}C}\left(Ab_X X \to X, E \xrightarrow{p} Y\right) = \mathrm{Hom}_C(X, E).$$

□

4 Limits and completeness

In this section, we study limits in $\mathbf{Mod}C$ and show that $\mathbf{Mod}C$ is complete if C is. Let's proceed in steps.

4.1 Limits in Beck modules

Proposition 4.1. *The forgetful functor $U : Ab(C) \to C$ creates limits, in the sense of [6, section 5.1].*

Proof. 1) Start with a diagram $\widetilde{F} : I \to Ab(C)$ whose underlying diagram $F := U\widetilde{F}$ in C has a limit. We will endow $\lim F$ with structure maps to produce a limit of $\widetilde{F}$. The structure maps of the objects in the diagram $\widetilde{F}$ can be expressed as natural transformations

$$\begin{aligned} F \times F &\xrightarrow{\mu} F \\ * &\xrightarrow{e} F \\ F &\xrightarrow{\iota} F, \end{aligned}$$

where $*$ is the terminal diagram (i.e. constant on the terminal object), and $F \times F$ is the composite

$$I \xrightarrow{(F,F)} C \times C \xrightarrow{\times} C.$$

In other words, we take the "objectwise" product, unit, and inverse structure maps. Applying the functor lim yields maps

$$\begin{aligned} \lim(F \times F) \cong \lim F \times \lim F &\xrightarrow{\lim \mu} \lim F \\ \lim * \cong * &\xrightarrow{\lim e} \lim F \\ \lim F &\xrightarrow{\lim \iota} \lim F. \end{aligned}$$

(Detail: We haven't assumed that C is complete, so technically there is no functor $\lim : C^I \to C$. We could work around this by restricting to the full subcategory of C^I of diagrams admitting a limit, on which the functor lim is defined. More explicitly, we can unwind the construction: A natural transformation $h : F \to G$ always induces $\lim h : \lim F \to \lim G$ whose associated cone on G is given by $\lim F \xrightarrow{\pi_i} F_i \xrightarrow{h_i} G_i$ for any index i in I.)

These form structure maps of an abelian group object, since applying lim to the condition diagrams of F yields condition diagrams for $\lim F$. Let us denote $\widetilde{\lim F} \in Ab(C)$ the object $\lim F$ equipped with these structure maps. By construction, $\widetilde{\lim F}$ comes with a cone on $\widetilde{F}$ which is a U-lift of $\lim F$ and its limiting cone on F.

2) Let us check that $\widetilde{\lim F}$ is the limit of $\widetilde{F}$ in $Ab(C)$. Given a cone $\widetilde{\psi} : \Delta\widetilde{Z} \to \widetilde{F}$, look at the underlying cone ψ; it has a unique map $\varphi Z \to \lim F$ associated to it. We need to check that φ is a map in $Ab(C)$. But by construction it is, since all the maps in the natural transformation $\widetilde{\psi}$ are in $Ab(C)$.

For example, consider the diagram

$$\begin{tikzcd}
Z \times Z \arrow[r, "{(\varphi,\varphi)}"] \arrow[d, "\mu_Z"'] & \lim F \times \lim F \arrow[r, "\cong"] \arrow[d, "\mu_{\lim F}"] & \lim(F \times F) \arrow[d, "\lim \mu_F"] \\
Z \arrow[r, "\varphi"'] & \lim F \arrow[r, equal] & \lim F.
\end{tikzcd}$$

It commutes iff the adjoint diagram commutes:

$$\begin{tikzcd}
\Delta Z \times \Delta Z = \Delta(Z \times Z) \arrow[r, "{(\psi,\psi)}"] \arrow[d, "\Delta\mu_Z"'] & F \times F \arrow[d, "\mu_F"] \\
\Delta Z \arrow[r, "\psi"'] & F.
\end{tikzcd}$$

This diagram does commute, since $\widetilde{\psi}$ was a natural transformation between I-diagrams in $Ab(C)$. Likewise for the unit and inverse structure maps. Hence we have $\widetilde{\lim F} = \lim \widetilde{F}$ in $Ab(C)$, as desired.

3) So far we've shown that U lifts limits, but there is more to creating limits. We need to show that there is a *unique* cone lifting $\lim F$ and its limiting cone $\pi : \Delta \lim F \to F$. Let L be such a lift, i.e. L is the underlying object $\lim F$ equipped with (possibly exotic) structure maps, and we have a lift $\widetilde{\pi} : \Delta L \to \widetilde{F}$ of the cone π. Saying that this cone $\widetilde{\pi}$ is a lift in $Ab(C)$ means that all the projection maps $\pi_i : \lim F \to F_i$ respect the structure maps. For example, the following diagram commutes:

$$\begin{tikzcd}
\lim F \times \lim F \arrow[r, "\mu"] \arrow[d, "\pi_i \times \pi_i"'] & \lim F \arrow[d, "\pi_i"] \\
F_i \times F_i \arrow[r, "\mu_i"'] & F_i.
\end{tikzcd}$$

Hence the i^{th} component of μ is $\mu_i(\pi_i \times \pi_i)$. But a map to a limit is uniquely determined by its components, hence μ is unique, and by the same argument, so are the unit and inverse structure maps. These structure maps are precisely the ones we used in part (1), i.e. L is $\widetilde{\lim F}$. Since U is faithful, the lift of the cone π is unique, and as we've seen in part (2), it is a limiting cone for $\widetilde{F}$. □

Corollary 4.2. *If C is complete, then so is $Ab(C)$, and $U : Ab(C) \to C$ preserves limits.*

Proof. Let $\widetilde{F} : I \to \mathbf{Mod}C$ be a diagram indexed by some (small) category I. Since C is complete, the underlying diagram $F := U\widetilde{F}$ has a limit, with limiting cone $\pi : \Delta \lim F \to F$. Since U creates limits, there is a unique U-lift $(L, \widetilde{\pi})$ of $(\lim F, \pi)$, and it's a limit in $\mathbf{Mod}C$. Thus U preserves limits, since they are essentially unique. □

Remark 4.3. Note that creating limits and preserving limits are distinct notions, neither one implying the other. The previous argument only shows that a limit-creating functor U must preserve limits of diagrams whose underlying diagram has a limit.

Example 4.4. (Preserving limits $\nRightarrow$ Creating limits) Consider the projection functor $C \times \mathcal{D} \to C$. Clearly it preserves limits, but (for general $\mathcal{D}$) it does not create them, or even lift them uniquely.

Example 4.5. (Creating limits $\nRightarrow$ Preserving limits) Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ be the category of natural numbers viewed as a totally ordered set, i.e. with a unique morphism from i to j iff $i \leq j$. Let $[n]$ be its full subcategory with objects $\{0, 1, \cdots, n\}$. Limits in $\mathbb{N}$ are as follows: the minimum number appearing in the diagram if the diagram is non-empty, and non-existent if the diagram is empty, i.e. there is no terminal object. Limits in $[n]$ are the same except there is a terminal object, namely n. Therefore, the inclusion functor $[n] \to \mathbb{N}$ does NOT preserve limits, but it does create them.

Remark 4.6. In fact, I believe U always preserves limits, without assuming C is complete.

Corollary 4.7. *If C is complete, then so is each category of Beck modules $\mathbf{Mod}_X$, for X an object in C.*

Proof. C being complete implies the slice category C/X is also complete [3, prop 2.16.3]. Hence $\mathbf{Mod}_X = Ab(C/X)$ is also complete. □

4.2 Changing the ground object

Proposition 4.8. *For any map $f : X \to Y$, the pullback functor $f^* : \mathbf{Mod}_Y \to \mathbf{Mod}_X$ preserves limits.*

Proof. We have the commutative diagram:

$$\begin{tikzcd} Ab(C/Y) \arrow[r, "f^*"] \arrow[d, "U_Y"'] & Ab(C/X) \arrow[d, "U_X"] \\ C/Y \arrow[r, "f^*"] & C/X. \end{tikzcd}$$

As seen above, U_Y preserves limits, and so does f^* (downstairs) since it's a right adjoint. Hence, if we start with a limit and its cone in $\mathbf{Mod}_Y$ we obtain:

$$\begin{aligned} U_X f^*(\lim F) &= f^* U_Y(\lim F) \\ &= \lim(f^* U_Y F). \end{aligned}$$

Since U_X creates limits, $f^*(\lim F)$ with its cone is the unique lift of $\lim(f^* U_Y F)$ with its cone, and it is itself a limit. In other words, we have $f^*(\lim F) = \lim f^* F$. □

Proposition 4.9. *If C is complete, then so is* $\mathbf{Mod}C$.

Proof. 1) Let $\widetilde{F} : I \to \mathbf{Mod}C$ be a diagram and denote $F := U\widetilde{F}$ the diagram of underlying objects in C. Since C is complete, it admits a limit $X := \lim F$. We're looking for a "limit module" over X; let's just pull back all the modules in $\widetilde{F}$ via the cone π. Having a diagram $\widetilde{F}$ in $\mathbf{Mod}C$ means that for every index $i \in I$, we have and object $\widetilde{F}_i \to F_i$ and for every map $\alpha : i \to j$ we have a map

$$\begin{tikzcd} \widetilde{F}_i \arrow[r] \arrow[d] & \widetilde{F}_j \arrow[d] \\ F_i \arrow[r, "F_\alpha"'] & F_j \end{tikzcd}$$

in $\mathbf{Mod}C$, which is the same as F_α and the associated map $\widetilde{F}_i \to F_\alpha^* \widetilde{F}_j$ in $\mathbf{Mod}_{F_i}$. Now, X has a limiting cone on F, i.e.

$$\begin{tikzcd} & X \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ F_i \arrow[rr, "F_\alpha"'] & & F_j \end{tikzcd}$$

Pulling back the modules over X via π, we obtain:

$$\begin{tikzcd}
\pi_i^*\widetilde{F_i} \arrow[rr] \arrow[dr] & & \pi_i^* F_\alpha^* \widetilde{F_j} \cong (F_\alpha \pi_i)^* \widetilde{F_j} = \pi_j^* \widetilde{F_j} \arrow[dl] \\
& X &
\end{tikzcd}$$

which defines an I-diagram $\pi^*\widetilde{F}$ in $\mathbf{Mod}_X$. By 4.7, $\mathbf{Mod}_X$ is complete, so we can take its limit $M := \lim \pi^*\widetilde{F}$.

2) Let us check that $M \to X$ and its cone $\widetilde{\pi}$ over $\widetilde{F}$ is a limit in $\mathbf{Mod}C$. Given a cone $\widetilde{\psi} : \Delta(\widetilde{Z} \to Z) \to \widetilde{F}$, look at the cone ψ of underlying objects and take its associated map $\varphi : Z \to X = \lim F$. We're looking for a map

$$\begin{tikzcd}
\widetilde{Z} \arrow[r, "\widetilde{\varphi}"] \arrow[d] & M \arrow[d] \\
Z \arrow[r, "\varphi"'] & X
\end{tikzcd}$$

that commutes with the cones, i.e. such that $\widetilde{\pi} \circ \widetilde{\varphi}$ is $\widetilde{\psi}$. This is the same as a map $\widetilde{Z} \to \varphi^* M$ in $\mathbf{Mod}_Z$ such that the cones $\widetilde{\pi} \circ \widetilde{\varphi}$ and $\widetilde{\psi}$ in $\mathbf{Mod}_Z$ over $\psi^*\widetilde{F}$ agree. There is a unique such map, because of the following:

$$\begin{aligned}
\varphi^* M &= \varphi^* \lim \pi^* \widetilde{F} \\
&= \lim \varphi^* \pi^* \widetilde{F} \quad \text{by 4.8} \\
&\cong \lim (\pi\varphi)^* \widetilde{F} \\
&= \lim \psi^* \widetilde{F}.
\end{aligned}$$

□

Question. Does $\mathbf{Mod}C$ inherit other nice properties from C? Here are some properties that would be of particular interest.

- Having all pushforwards;
- Having all abelianizations;
- Beck modules over any object form an abelian category;
- Cocompleteness;

- Regularity;
- Exactness.

I suspect the answer is yes in all cases.

5 Representability

In the examples of section 2, we have seen that the category of Beck modules over an object X is sometimes naturally equivalent to a category of functors into **Ab**, i.e. presheaves of abelian groups. This raises the question: Is there a (covariant) pseudofunctor $I_{(-)}$ making the following diagram commute up to natural isomorphism?

$$\begin{tikzcd} & \mathbf{Cat} \arrow[dr, "\mathrm{Fun}(-,\mathbf{Ab})"] & \\ C \arrow[ur, dashed, "I_{(-)}"] \arrow[rr, "\mathbf{Mod}_{(-)}"'] & & \mathbf{AbCat} \end{tikzcd}$$

Definition 5.1. The category C has **representable Beck modules** if there exists such a pseudofunctor $I_{(-)}$, called an **indexing functor**, and the category I_X, satisfying $\mathbf{Mod}_X \cong \mathbf{Ab}^{I_X}$ is called an **indexing category** for the object X of C.

Example 5.2. $C = \mathbf{Set}$ has representable Beck modules, taking the indexing category I_X of a set X to be the discrete small category X.

Example 5.3. $C = \mathbf{Gp}$ has representable Beck modules, taking the indexing category I_G of a group G to be the one-object groupoid G.

Example 5.4. $C = \mathbf{Ab}$ has representable Beck modules, taking the indexing category I_A of an abelian group A to be the trivial category $\{*\}$ with one object and its identity map.

Example 5.5. $C = \mathbf{Mon}$, the category of monoids, has representable Beck modules, taking the indexing category I_M of a monoid M to be its factorization category $Fac(M)$.

Non-example 5.6. $C = \mathbf{ComRing}$, the category of commutative (associative, unital) rings, does NOT have representable Beck modules. For a commutative ring R, the category $\mathbf{Mod}_R$ of Beck modules over it is the usual category of R-modules. Taking $R = \mathbb{F}_p$ the field with p elements, then $\mathbf{Mod}_R$ is the category of $\mathbb{F}_p$-vector spaces, which itself is not of the form $\mathbf{Ab}^I$ for any category I. Indeed, fixing an index (object) i of I, consider the (covariant) functor:

$$I \xrightarrow{\mathrm{Hom}_I(i,-)} \mathbf{Set} \xrightarrow{Free} \mathbf{Ab}$$

This is an object of $\mathbf{Ab}^I$ whose identify endomorphism has infinite additive order. This can't happen in $\mathbb{F}_p$-vector spaces, where every map is p-torsion.

Question. When does C have representable Beck modules? This question can be broken down into two.

1. When is an abelian category $\mathcal{A}$ equivalent to a functor category $\mathbf{Ab}^I$?

2. Assuming C is such that for every object X, the category $\mathbf{Mod}_X$ is equivalent to a functor category $\mathbf{Ab}^{I_X}$ for some (small) category I_X, when does C have representable Beck modules? In other words, can we make the indexing categories I_X and equivalences $\mathbf{Mod}_X \cong \mathbf{Ab}^{I_X}$ natural in X?

Question. Does C have some nice features if it has representable Beck modules? In other words, is it an interesting property?

It is more common to look for a ring R (or ringoid, a.k.a. preadditive category) such that the abelian category $\mathcal{A}$ is the category of modules over R (additive functors $R \to \mathbf{Ab}$).

6 Analogy with the tangent bundle

Here's a far-fetched idea: Does the fibered category $\mathbf{Mod}C$ deserve to be thought of as the "tangent bundle" of C? The tangent bundle of a smooth manifold M gives over each point x the tangent space T_xM, which can be thought of as the best linear approximation of M around x. Heuristically, let's think of C/X as a neighborhood of X, in other words an object $Y \xrightarrow{f} X$ is a point "close" to X. Now let's think of abelian group objects as providing the "best approximation" by an additive category. Then $\mathbf{Mod}_X$ is a "best linear approximation" of C around the object X.

Here's one way in which the analogy is not completely silly. Notice that if V is a smooth manifold that happens to be a vector space (i.e. Euclidean space), then for every point $x \in V$, we have $T_xV \cong V$. The same holds for Beck modules.

Lemma 6.1. *Let I be any category and let $\mathcal{A}$ denote the abelian category $\mathbf{Ab}^I$. Then for every object F of $\mathcal{A}$, we have an equivalence of abelian categories:*

$$\mathbf{Mod}_F = Ab(\mathcal{A}/M) \cong \mathcal{A}.$$

The equivalence associates to a module $E \xrightarrow{p} F$ the object $\ker p$, *and to an object K of $\mathcal{A}$, the module $F \oplus K \to F$, with structure maps given by the "objectwise" addition in K.*

Proof. Let's look at one node of the diagram at a time. For an index i in I, consider the functor $i^* : \mathbf{Ab}^I \to \mathbf{Ab}$ that extracts the abelian group at index i, i.e. evaluates a functor in $\mathbf{Ab}^I$ at i. This is the restriction functor along the inclusion of the point category $i : \{*\} \to I$ that selects the object i. Since i^* preserves limits, it induces a functor on Beck modules. By our knowledge of Beck modules in $\mathbf{Ab}$ (proposition 2.7), we know that the Beck module $E \xrightarrow{p} F$, which looks like

$$\begin{tikzcd} E_i \arrow[r, "E_\alpha"] \arrow[d, "p_i"'] & E_j \arrow[d, "p_j"] \\ F_i \arrow[r, "F_\alpha"] & F_j \end{tikzcd}$$

(where $\alpha : i \to j$ is a typical map in I) is actually of the form

$$\begin{tikzcd} K_i \arrow[r, "K_\alpha"] \arrow[d] & K_j \arrow[d] \\ F_i \oplus K_i \arrow[r, "E_\alpha"] \arrow[d, "p_i"'] & F_j \oplus K_j \arrow[d, "p_j"] \\ F_i \arrow[r, "F_\alpha"] & F_j \end{tikzcd}$$

where K_i is $\ker p_i$ and K_α is the restriction of E_α to K_i. Moreover, since $p : F \oplus K \to F$ is a map in $\mathcal{A}$, i.e. a natural transformation, and so is the zero section $e : F \to F \oplus K$, we know that the map E_α is actually $F_\alpha \oplus K_\alpha$, or in matrix form $\begin{bmatrix} F_\alpha & 0 \\ 0 & K_\alpha \end{bmatrix}$. The upper row comes from p, and the lower-left corner comes from e. This determines completely the structure maps, which must be the objectwise addition, zero, and negative in each abelian group K_i. The only additional information is that these structure maps are maps in $\mathcal{A}/F$. Writing down these conditions explicitly, they say that K_α must be a group map, which is automatic. In other words, there is no constraint on K, any object of $\mathcal{A}$ will do.

Now look at the correspondence described in the statement. The composite $\mathcal{A} \to \mathbf{Mod}_F \to \mathcal{A}$ is the identify functor. The composite $\mathbf{Mod}_F \to \mathcal{A} \to \mathbf{Mod}_F$ sends $E \xrightarrow{p} F$ to $F \oplus \ker p \to F$, which is a natural iso in $\mathcal{A}/F$ since the Beck module comes equipped with the data of the splitting (the unit map). By the argument above, it is a natural iso in $\mathbf{Mod}_F$, i.e. it recovers the Beck module structure. Finally, note that both directions of the correspondence are additive functors, so we obtain an equivalence of abelian categories $\mathbf{Mod}_F \cong \mathcal{A}$. □

Proposition 6.2. *Let $\mathcal{A}$ be an abelian category. Then for every object M of $\mathcal{A}$, we have an equivalence of abelian categories:*

$$\mathbf{Mod}_M = Ab(\mathcal{A}/M) \cong \mathcal{A}.$$

The equivalence associates to a module $E \xrightarrow{p} M$ the object $\ker p$*, and to an object K of $\mathcal{A}$, the module $M \oplus K \to M$, with the following structure maps:*

- *Multiplication $\mu : M \oplus K \oplus K \to M \oplus K$ is given by* $\begin{bmatrix} \mathrm{id}_M & 0 & 0 \\ 0 & \mathrm{id}_K & \mathrm{id}_K \end{bmatrix}$;
- *Unit $e : M \to M \oplus K$ is given by inclusion, that is* $\begin{bmatrix} \mathrm{id}_M \\ 0 \end{bmatrix}$;
- *Inverse $\iota : M \oplus K \to M \oplus K$ is given by* $\begin{bmatrix} \mathrm{id}_M & 0 \\ 0 & -\mathrm{id}_K \end{bmatrix}$.

Proof. Essentially follows from the splitting lemma, the Yoneda embedding and the lemma above. Recall the Yoneda embedding:

$$\begin{aligned} Y : \mathcal{A} &\to \mathbf{Ab}^{\mathcal{A}^{op}} \\ B &\mapsto \mathrm{Hom}_{\mathcal{A}}(-, B) \end{aligned}$$

which is full, faithful, and left exact. Since it's left exact, it preserves small limits and hence induces a functor on Beck modules. Starting with a Beck module $E \xrightarrow{p} M$ in $\mathcal{A}$, we get a Beck module $Y(E) \xrightarrow{Y(p)} Y(M)$ in $\mathbf{Ab}^{\mathcal{A}^{op}}$. By the lemma above, its structure maps must be the obvious ones on $\ker Y(p)$ (objectwise addition, zero, and inverse). Since Y is faithful, this determines the structure maps for $E \xrightarrow{p} M$.

Moreover, we have $\ker Y(p) = Y(\ker p)$, and by the splitting lemma, $E \xrightarrow{p} M$ is canonically isomorphic to $M \oplus \ker p \to M$ in $\mathcal{A}$. Put structure maps on the latter by the formulas in the statement. Notice that Y sends all those maps to the structure maps of $Y(M) \oplus \ker Y(p) \to Y(M)$ (we used the fact that Y is additive for the inverse structure map ι). Hence those candidate structure maps ARE those of $E \xrightarrow{p} M$. (Incidentally, this shows that the formulas in the statement do define a Beck module, although one can easily check it directly.)

We can conclude $\mathbf{Mod}_M \cong \mathcal{A}$ exactly as in the lemma. $\square$

Question.

1. Can we push the analogy further? More precisely, we're looking for a universal property satisfied by the tangent bundle of a manifold, and a universal property satisfied by the fibered category of Beck modules.
2. Is $Ab(C) \to C$ terminal among additive categories equipped with a faithful, limit-creating functor to C?

7 Modules over a simplicial object

In this section, we present another context in which $\mathbf{Mod}C$ appears naturally. Quillen's notion of cohomology involves taking simplicial resolutions of objects, which led him to study simplicial modules over a simplicial object, such as a simplicial ring [7, II.6]. Let us first describe modules over a simplicial set.

Proposition 7.1. *Let X be a simplicial set. Then a Beck module $E \xrightarrow{p} X$ over X is the data of a Beck module $p_n : E_n \to X_n$ (i.e. a bundle of abelian groups) in each simplicial degree, such that the face and degeneracies of E cover those of X and respect the abelian group object structure maps, i.e. are fiberwise maps of abelian group.*

Proof. The "n^{th} degree" functor $s\mathbf{Set} \to \mathbf{Set}$ preserves limits, since it is the restriction along $n \hookrightarrow \Delta^{\mathrm{op}}$. Hence it induces a functor on Beck modules and we get a Beck module $p_n : E_n \to X_n$ in each simplicial degree. This determines the constituent sets of E, the map p, and the abelian group structure maps of E. The remaining conditions are that $p : E \to X$ is a map in $s\mathbf{Set}$ and the structure maps are maps in $s\mathbf{Set}/X$. Those are exactly the conditions mentioned in the claim. $\square$

In fact, there's nothing special about $\mathbf{Set}$ and Δ^{op}. Instead of $\mathbf{Set}$, take C a category with finite limits, and instead of Δ^{op}, take I any (small) category. Recall the notation $U : \mathbf{Mod}C \to C$ for the "base object" forgetful functor. Consider the category $C^I = \mathrm{Fun}(I, C)$ of I-diagrams in C.

Proposition 7.2. *Given a diagram $F : I \to C$, the category $\mathbf{Mod}_F = Ab(C^I/F)$ of Beck modules over F is the category of I-diagrams like this: the i^{th} entry is a Beck module over $F(i)$ and maps respect the structures of Beck modules (i.e. are maps in $\mathbf{Mod}C$).*

In other words, $\mathbf{Mod}_F$ is the category of I-diagrams in $\mathbf{Mod}C$ whose base diagram is F and where morphisms fix the base, i.e. the fiber over F of the forgetful functor:

$$U^I : \mathbf{Mod}C^I \to C^I.$$

For $I = \Delta^{\mathrm{op}}$ and $C = \mathbf{ComRing}$, we get simplicial modules over a simplicial ring, as studied by Quillen.

References

[1] M. Barr. *Acyclic Models*. Number 17 in CRM Monograph Series. American Mathematical Society, 2002.

[2] H.J. Baues. *Combinatorial homotopy and* 4*-dimensional complexes*. Walter de Gruyter, 1991.

[3] F. Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*. Number 50 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[4] W.G. Dwyer. Homotopy theory and classifying spaces. Minicourse taught at the University of Copenhagen, June 2008.

[5] P.G. Goerss and K. Schemmerhorn. Model categories and simplicial methods. *Contemporary Mathematics*, 436:3–49, 2007.

[6] S. MacLane. *Categories for the Working Mathematician*. Number 5 in Graduate Texts in Mathematics. Springer, second edition, 1998.

[7] D.G. Quillen. *Homotopical Algebra*. Number 43 in Lecture Notes in Mathematics. Springer, 1967.

[8] A. Vistoli. Notes on Grothendieck topologies, fibered categories and descent theory. In B. Fantechi et al., editor, *Fundamental Algebraic Geometry: Grothendieck's FGA explained*, number 123 in Mathematical surveys and monographs. AMS, 2006, rev. 2007.