The algebra of tertiary cohomology operations

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Outline

1. Background
2. d_2 via secondary resolutions
3. d_r and higher null-homotopies
4. In dimension 2

Classical Adams spectral sequence

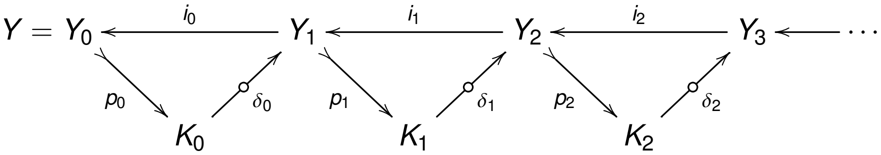
X, Y finite spectra, $H^*X := H^*(X; \mathbb{F}_p)$. The classical Adams spectral sequence is

$$E_2^{s,t} = \mathrm{Ext}_{\mathfrak{A}}^{s,t}(H^*Y, H^*X) \Rightarrow [\Sigma^{t-s}X, Y_p^{\wedge}]$$

where $\mathfrak{A} = H\mathbb{F}_p^* H\mathbb{F}_p$ is the mod p Steenrod algebra. In particular, $X = Y = S^0$ yields

$$E_2^{s,t} = \mathrm{Ext}_{\mathfrak{A}}^{s,t}\left(\mathbb{F}_p, \mathbb{F}_p\right) \Rightarrow \pi_{t-s}^S(S^0)_p^{\wedge}.$$

Adams resolution of Y:



where:

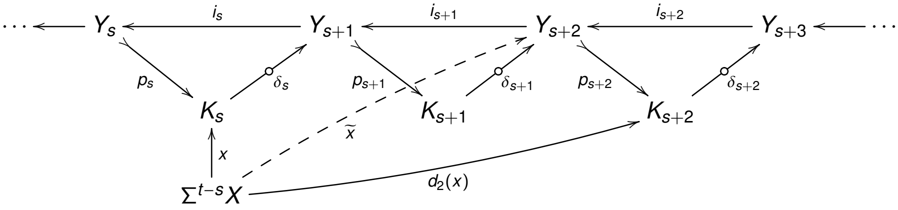
- $K_s = \prod_i \Sigma^{n_i} H\mathbb{F}_p$ degreewise finite, bounded below.
- $H^* i_s = 0$.
- Fiber sequences $Y_{s+1} \xrightarrow{i_s} Y_s \xrightarrow{p_s} K_s$.

$\leftrightsquigarrow$ Injective resolution of Y:

$$0 \longrightarrow Y \xrightarrow{p_0} K_0 \xrightarrow{d_1} \Sigma K_1 \xrightarrow{d_1} \Sigma^2 K_2 \xrightarrow{d_1} \cdots$$

Differentials as higher cohomology operations

Take $[x] \in E_2^{s,t}$ represented by a cycle $x \in E_1^{s,t} = [\Sigma^{t-s} X, K_s]$. Recall that $d_2[x] \in E_2^{s+2,t+1}$ is obtained as:



$d_2(x)$ is a certain element of the Toda bracket $\langle \Sigma d_1, d_1, x \rangle$.
This is a *secondary* cohomology operation.

Differentials as higher cohomology operations (cont'd)

d_r is given by r^{th} order cohomology operations [Maunder 1964].

Idea: The resolution encodes *coherent* witnesses of the equations $d_1 d_1 = 0$.

Different approaches:

- *Triangulated*: Witnesses are lifts to fibers or extensions to cofibers. [Christensen–F. 2015]
- *Topologically enriched*: Witnesses are null-homotopies, i.e., paths (and cubes) to zero in mapping spaces. [Baues–Jibladze 2006, 2010, Baues–Blanc 2015]

Goals

Ultimate goal

Compute d_3.

Actual goals

- Describe the algebraic structure involved in computing d_3.
  Ref: 2-track algebras and the Adams spectral sequence. *To appear*.
- Revisit and streamline the work of Baues on secondary operations.
  Ref: The DG-algebra of secondary cohomology operations. *In preparation*.
- Use a similar strategy to tackle tertiary operations.
  Ref: Work in progress...

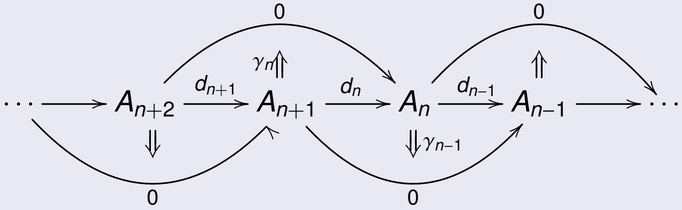
Notation

Notation

- $\mathbf{Top}_*$:= *pointed topological spaces. Basepoints are denoted* $0 \in X$*. Enrichment means in* $(\mathbf{Top}_*, \wedge)$*.*
- $\mathbf{Spec}$:= *topologically enriched category of spectra.*
- $\Pi_{(1)}X$:= *fundamental groupoid of a space* X*.*
- *A **pointed** groupoid* G *is equipped with a base object, denoted* $0 \in G_0$*.*
- *Composition in a groupoid is denoted* $\Box$*, with identity* $\mathrm{id}^{\Box}_x : x \to x$ *and inverse of* $f : x \to y$ *denoted* $f^{\boxminus} : y \to x$*.*

Definition

*Let $\mathcal{G}$ be a category enriched in pointed groupoids. A **secondary pre-chain complex** (A, d, γ) in $\mathcal{G}$ is:*



(A, d, γ) *is a **secondary chain complex** if moreover for each $n \in \mathbb{Z}$:*

$$d_{n-1}\gamma_n = \gamma_{n-1}d_{n+1} : d_{n-1}d_nd_{n+1} \Rightarrow 0.$$

Secondary chain complexes (cont'd)

In other words:

$$(\gamma_{n-1} d_{n+1}) \,\square\, (d_{n-1}\gamma_n)^{\boxminus} = \mathrm{id}_0^{\square} : 0 \Rightarrow 0$$

in the groupoid $\mathcal{G}(A_{n+2}, A_{n-1})$. Note that this track is in the Toda bracket

$$\langle d_{n+1}, d_n, d_{n-1} \rangle \subseteq \pi_1 \mathcal{G}\left(A_{n+2}, A_{n-1}\right).$$

These Toda brackets vanish *coherently*.

d_2 via secondary resolutions

Adams spectral sequence abutting to $[X, Y]$, with

$$E_2^{s,t} = \operatorname{Ext}_{\mathfrak{A}}^{s,t}(H^* Y, H^* X) .$$

Free resolution of $H^* Y$ as $\mathfrak{A}$-module:

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H^* Y.$$

Realize topologically as the cohomology of a diagram in the stable homotopy category $\pi_0\mathbf{Spec}$:

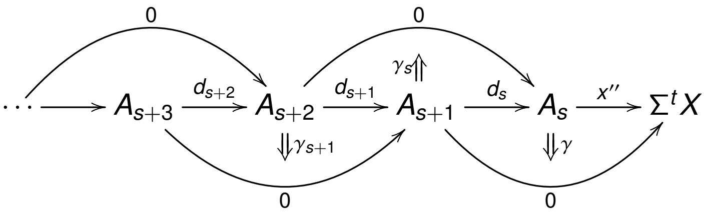
$$\cdots \longleftarrow A_2 \xleftarrow{d_1} A_1 \xleftarrow{d_0} A_0 \xleftarrow{\epsilon} A_{-1} = Y$$

with $A_s \simeq \prod_i \Sigma^{n_i} H\mathbb{F}_p$ (for $s \geq 0$) and $H^* A_s \cong F_s$.
Since we prefer chain complexes to cochain complexes, work in the opposite category $\pi_0\mathbf{Spec}^{\mathrm{op}}$:

$$\cdots \longrightarrow A_2 \xrightarrow{d_1} A_1 \xrightarrow{d_0} A_0 \xrightarrow{\epsilon} A_{-1} = Y.$$

d_2 via secondary resolutions (cont'd)

- Lift this resolution $A_\bullet \to Y$ in $\pi_0\mathbf{Spec}^{\mathrm{op}}$ to a secondary resolution (A, d, γ) in $\Pi_{(1)}\mathbf{Spec}^{\mathrm{op}}$.
- Start with a class $x \in E_2^{s,t} = \mathrm{Ext}_{\mathfrak{A}}^{s,t}(H^*Y, H^*X)$ represented by a cocycle $x' \colon F_s \to \Sigma^t H^* X$.
- Realize x' as the cohomology of a map $x'' \colon A_s \to \Sigma^t X$ in $\mathbf{Spec}^{\mathrm{op}}$.
- The equation $x' d_s = 0$ means that $x'' d_s$ is null-homotopic. Choose a null-homotopy $\gamma \colon x'' d_s \Rightarrow 0$.



d_2 via secondary resolutions (cont'd)

Theorem (Baues–Jibladze 2006)

The obstruction

$$\gamma d_{s+1} \square (x'' \gamma_s)^{\boxminus} \in \pi_1 \mathbf{Spec}^{\mathrm{op}}(A_{s+2}, \Sigma^t X) = \pi_0 \mathbf{Spec}^{\mathrm{op}}(A_{s+2}, \Sigma^{t+1} X)$$

is a (co)cycle and does not depend on the choices, up to (co)boundaries, and thus defines an element:

$$d_{(2)}(x) \in \mathrm{Ext}_{\mathfrak{A}}^{s+2,t+1}(H^* Y, H^* X).$$

This is the Adams differential d_2.

Generalizing to d_r

Question

How to generalize this constuction to higher differentials d_r?

An answer

Use higher dimensional null-homotopies, i.e., cubes to zero. Describe how the cubes paste together.

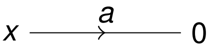
Cubes in a space

Definition

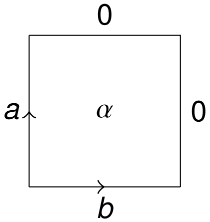
Let X be a pointed space.

- *An n-**cube** in X is a map $a\colon I^n \to X$.*
- *An n-**track** in X is the homotopy class $\{a\}$ rel ∂I^n.*
- *A **left** n-**cube** in X is an n-cube a satisfying $a(t_1, \dots, t_n) = 0$ whenever a coordinate $t_k = 1$.*
- *A **left** n-**track** is the homotopy class rel ∂I^n of a left n-cube.*

A left 1-cube:



A left 2-cube:



Left cubical sets

Definition

- *The **singular cubical set** $S^{\square}(X)$ of X has $S^{\square}(X)_n = \mathrm{Map}(I^n, X)$ and restriction maps along the $2n$ faces (and degeneracies).*
- *The **left cubical set** $\mathrm{nul}(X)$ of X has $\mathrm{nul}(X)_n = \{$left n-cubes in $X\}$ and restriction maps along the n "left" faces (no degeneracies anymore).*
- *The **left n-cubical set** $\mathrm{Nul}_n(X)$ of X has*

$$\mathrm{Nul}_n(X)_k = \begin{cases} \text{left } k\text{-cubes in } X & \text{if } k < n \\ \text{left } n\text{-tracks in } X & \text{if } k = n \end{cases}$$

Product of cubes

Let $\mathcal{C}$ be **Top**$_*$-enriched, with composition

$$\mathcal{C}(B, C) \wedge \mathcal{C}(A, B) \xrightarrow{\mu} \mathcal{C}(A, C).$$

Given cubes $b \colon I^m \to \mathcal{C}(B, C)$ and $a \colon I^n \to \mathcal{C}(A, B)$ the **$\otimes$-composition** of b and a is the $(m + n)$-cube

$$b \otimes a \colon I^{m+n} = I^m \times I^n \xrightarrow{b \times a} \mathcal{C}(B, C) \times \mathcal{C}(A, B) \xrightarrow{\mu} \mathcal{C}(A, B).$$

The $\otimes$-composition of left cubes is a left cube. Take the left n-cubical set of each mapping space

$$(\mathrm{Nul}_n \mathcal{C})(A, B) := \mathrm{Nul}_n (\mathcal{C}(A, C)) .$$

This makes $\mathrm{Nul}\, \mathcal{C}$ into an n-graded category.

Cubical balls

“Definition”

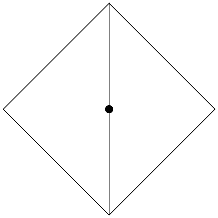
A **left n-cubical ball** in a pointed space X consists of left n-tracks in X that paste together into a disk D^n whose boundary sphere ∂D^n consists of the “right” sides, all mapped to the basepoint of X. This defines a homotopy class of map $(D^n, \partial D^n) \to (X, 0)$, i.e., an element of $\pi_n X$.

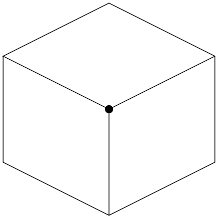
“Definition”

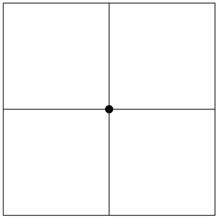
An **algebra of left n-cubical balls** is roughly the algebraic structure found in $\mathrm{Nul}_n \mathcal{C}$, where $\mathcal{C}$ is $\mathbf{Top}_*$-enriched.
The important piece of data is an *obstruction operator*, which gives the value in $\pi_n X$ of left n-cubical balls.

Cubical balls (cont'd)

Some left cubical balls of dimension 2:







Obtaining d_r

Back to our finite spectra X and Y. Take the $\mathbf{Top}_*$-enriched category of GEM spectra, along with mapping spaces from X or Y into GEM spectra. Let $\mathcal{C}$ be the opposite category.

Theorem (Baues–Blanc 2015)

The algebra of left n-cubical balls $\mathrm{Nul}_n \mathcal{C}$ *determines the Adams differential* d_{n+1}.

Idea: Lift the $\mathfrak{A}$-module resolution of $H^* Y$ to a higher order resolution in $\mathrm{Nul}_n \mathcal{C}$. Proceed as before.

Combinatorial difficulty

Problem

The combinatorics of cubical balls becomes messy in higher dimensions. Cubical balls of dimension n correspond to triangulations of the sphere $\partial D^n = S^{n-1}$.

Good news

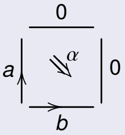
No complication when $n = 2$.

Fundamental 2-track groupoid

Specialize [Baues–Blanc] to $n = 2$.

Definition

Let X be a pointed space. Consider the groupoid $\Pi_{(2)}(X)$ with objects the left 1-cubes in X, and morphisms the left 2-tracks in X:



*The **fundamental 2-track groupoid** of X is the pair of pointed groupoids*

$$\Pi_{(1,2)}(X) := \big(\Pi_{(1)}(X), \Pi_{(2)}(X)\big)$$

+ a bit of extra structure.

2-track groupoids

Definition

A 2*-**track groupoid** is a pair of pointed groupoids*

$$G = \left(G_{(1)}, G_{(2)}\right)$$

with:

1. $G_{(2)}$ *is equipped with isomorphisms* $\psi_a \colon \operatorname{Aut}(a) \xrightarrow{\simeq} \operatorname{Aut}(0)$ *for each object a, which commute with change of basepoint isomorphisms.*
2. *A bijection between components of* $G_{(2)}$ *and morphisms to* 0 *in* $G_{(1)}$*.*

Weak equivalences

Definition

*The **homotopy groups** of a 2-track groupoid G are*

$$\pi_0 G = \mathrm{Comp}\, G_{(1)}$$
$$\pi_1 G = \mathrm{Aut}_{G_{(1)}}(0)$$
$$\pi_2 G = \mathrm{Aut}_{G_{(2)}}(0).$$

*A morphism $F \colon G \to G'$ of 2-track groupoids is a **weak equivalence** if it induces isomorphisms on homotopy groups.*

Note: $\pi_i \Pi_{(1,2)}(X) = \pi_i X$.

Homotopy 2-types

The fundamental 2-track groupoid $\Pi_{(1,2)}(X)$ remembers much less than the homotopy 2-type of X.

Proposition

Connected 2-track groupoids G and G′ are weakly equivalent if and only if they have isomorphic homotopy groups π_1 and π_2.

2-track algebras

"Definition"

A 2**-track algebra** is roughly the algebraic structure found in $\Pi_{(1,2)}C$ for a **Top**$_*$-enriched category C.

Can define tertiary (pre-)chain complexes in a 2-track algebra.

Obtaining d_3

Take C as before.

Theorem (Baues–F. 2015)

1. *The 2-track algebra $\Pi_{(1,2)}C$ determines the Adams differential d_3.*
2. *This construction of d_3 depends only on the weak equivalence class of the 2-track algebra.*

Idea: The obstruction operator in $\Pi_{(1,2)}C$ is given by concatenating left 2-tracks.

Thank you!