Non-realizable 2-stage Π-algebras

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Outline

π_*-modules and Π-algebras

- X = spectrum or space
  $\pi_* X$ = graded group with (primary) **homotopy operations**.
- X = spectrum
  $\pi_* X$ is a $(\pi_* S)$-module, where $S :=$ sphere spectrum.
  $\alpha \in \pi_k S \rightsquigarrow$ Precomposition $\alpha^* : \pi_n X \to \pi_{n+k} X$

$$S^{n+k} \xrightarrow{\alpha} S^n \xrightarrow{x} X$$

- X = connected space
  $\pi_* X$ is a Π-algebra.
  $\alpha \in \pi_{n+k} S^n \rightsquigarrow$ Precomposition $\alpha^* : \pi_n X \to \pi_{n+k} X$

$$S^{n+k} \xrightarrow{\alpha} S^n \xrightarrow{x} X$$

Also π_1-action and Whitehead products $\pi_p \times \pi_q \to \pi_{p+q-1}$.

Π-algebra $\approx$ graded group with additional structure which looks like the homotopy groups of a space.

Definition

- Π := full subcategory of the homotopy category of pointed spaces consisting of finite wedges of spheres $\bigvee S^{n_i}$, $n_i \geq 1$.
- Π-**algebra** := product-preserving functor $\pi\colon \Pi^{\mathrm{op}} \to \mathbf{Set}$.

Example

$\pi_* X = [-, X]_*$ for a pointed space X.

Notation: Write $\pi_n := \pi(S^n)$.

Realization and classification

Realization Problem

Given a Π-algebra π, is there a space X satisfying $\pi_* X \simeq \pi$ as Π-algebras?

Classification Problem

If π is realizable, can we classify all realizations?

This project only deals with the **realization problem**.

Remark

A Π-algebra in the stable range is realizable if and only if the corresponding π_*-module is realizable.

2-stage case

Simplest Π-algebras: Only one non-trivial group π_n.
Answer: Always realizable (uniquely), by an Eilenberg-Maclane space $K(\pi_n, n)$.

Next simplest case: Only 2 non-trivial groups π_n, π_{n+k}. Assume $n \geq 2$.
Answer: **Not** always realizable...

Goals

1. Find necessary and sufficient conditions for a 2-stage Π-algebra to be realizable.
2. Provide non-realizable examples.

Warm-up

Case $k = 1 \rightsquigarrow$ Always realizable (classic).
Case $k = 2 \rightsquigarrow$ Always realizable (a bit of work).

Homotopy operation functors

Reference: Baues, Goerss (2000).

Idea: Encode the Π-algebra data inductively.

Notation

$\mathbf{\Pi Alg}_n^k$:= full subcategory consisting of Π-algebras concentrated in degrees $n, n+1, \ldots, n+k$.

Data consists of abelian groups and structure maps

$$\begin{aligned} &\pi_n \\ \eta_1 &: \Gamma_n^1(\pi_n) \to \pi_{n+1} \\ \eta_2 &: \Gamma_n^2(\pi_n, \eta_1) \to \pi_{n+2} \\ &\ldots \\ \eta_k &: \Gamma_n^k(\pi_n, \eta_1, \ldots, \eta_{k-1}) \to \pi_{n+k}. \end{aligned}$$

Homotopy operation functors

Postnikov truncation $\mathbf{\Pi Alg}_n^k \to \mathbf{\Pi Alg}_n^{k-1}$ has a left adjoint L.
Γ_n^k is the composite

$$\mathbf{\Pi Alg}_n^{k-1} \xrightarrow{L} \mathbf{\Pi Alg}_n^k \xrightarrow{(n+k)^{\text{th}} \text{ group}} \mathbf{Ab}.$$

Example

$$\Gamma_n^1(\pi_n) = \begin{cases} \Gamma(\pi_n) & \text{for } n = 2 \\ \pi_n \otimes_{\mathbb{Z}} \mathbb{Z}/2 & \text{for } n \geq 3. \end{cases}$$

and $\eta_1 \colon \Gamma_n^1(\pi_n) \to \pi_{n+1}$ is precomposition by the Hopf map $\eta \colon S^{n+1} \to S^n$.

Homotopy operation functors

A 2-stage Π-algebra consists of the data

$$\pi_n$$

$$\eta_k : \widetilde{\Gamma_n^k}(\pi_n) := \Gamma_n^k(\pi_n, 0, \ldots, 0) \to \pi_{n+k}.$$

Notation

$Q_{k,n} :=$ indecomposables of $\pi_{n+k}(S^n)$
In the stable range $k \leq n-2$, we have $Q_{k,n} = Q_k^S$
($Q_*^S :=$ indecomposables of the graded ring π_*^S).

Proposition

Assuming $k \neq n-1$, we have

$$\widetilde{\Gamma_n^k}(\pi_n) = \pi_n \otimes_{\mathbb{Z}} Q_{k,n}.$$

In particular, in the stable range we have $\widetilde{\Gamma_n^k}(\pi_n) = \pi_n \otimes_{\mathbb{Z}} Q_k^S$.

Whitehead's exact sequence

$$\ldots \to H_{n+1}X \xrightarrow{b} \Gamma_n X \xrightarrow{i} \pi_n X \xrightarrow{h} H_n X \xrightarrow{b} \Gamma_{n-1}X \to \ldots$$

$h =$ Hurewicz map

$\Gamma_i X = \operatorname{im}\left(\pi_i X^{(i-1)} \to \pi_i X^{(i)}\right)$

Natural transformation γ:

$$\begin{array}{ccc} \Gamma_n^k(\pi_n, \eta_1, \ldots, \eta_{k-1}) & & \\ \downarrow \gamma & \searrow^{\eta_k} & \\ \Gamma_{n+k}X & \xrightarrow{\quad i \quad} & \pi_{n+k}X \end{array}$$

Main result: Criterion for realizability

Theorem (Baues,F.)

The 2-stage Π-algebra given by $\eta_k : \widetilde{\Gamma_n^k}(\pi_n) \to \pi_{n+k}$ is realizable if and only if the map η_k factors through the map $\gamma_{K(\pi_n,n)}$.

$$\begin{array}{ccc} & & \Gamma_{n+k}K(\pi_n,n) \\ & \overset{\gamma_{K(\pi_n,n)}}{\nearrow} & \downarrow \\ \widetilde{\Gamma_n^k}(\pi_n) & \xrightarrow[\eta_k]{} & \pi_{n+k} \end{array}$$

Key ingredient: Theorem on the realizibality of the Hurewicz map [Baues].

Remark: $\Gamma_{n+k}K(\pi_n,n) \cong H_{n+k+1}K(\pi_n,n)$.

Main result

Corollary

Fix $n \geq 2$ and $k \geq 1$. Then an abelian group π_n has the property that "every Π-algebra concentrated in degrees $n, n+k$ with prescribed group π_n is realizable" if and only if the map

$$\gamma_{K(\pi_n,n)} : \widetilde{\Gamma_n^k}(\pi_n) \to \Gamma_{n+k} K(\pi_n, n)$$

is split injective.

A simplification

In the stable range ($k \leq n - 2$), the map $\gamma_{K(\pi_n,n)}$ becomes

$$\begin{aligned} \gamma_{K(\pi_n,n)} : \pi_n \otimes_{\mathbb{Z}} Q_k^S &\to H_{n+k+1} K(\pi_n, n) \\ &\cong (H\mathbb{Z})_{k+1}(H\pi_n) \\ &\cong (H\pi_n)_{k+1}(H\mathbb{Z}). \end{aligned}$$

Proposition

$\gamma_{K(\pi_n,n)}$ *factors through the summand* $\pi_n \otimes_{\mathbb{Z}} H\mathbb{Z}_{k+1}H\mathbb{Z} \hookrightarrow (H\pi_n)_{k+1}H\mathbb{Z}$.

Upshot

Take $\pi_n = \mathbb{Z}$ and study $\gamma_{K(\mathbb{Z},n)} : Q_k^S \to H\mathbb{Z}_{k+1}H\mathbb{Z}$.

Non-realizable example

First few stable homotopy groups of spheres π_*^S and their indecomposables Q_*^S.

k	π_k^S	Q_k^S
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2\langle\eta\rangle$	$\mathbb{Z}/2\langle\eta\rangle$
2	$\mathbb{Z}/2\left\langle\eta^2\right\rangle$	0
3	$\mathbb{Z}/24 \simeq \mathbb{Z}/8\langle\nu\rangle \oplus \mathbb{Z}/3\langle\alpha\rangle$	$\mathbb{Z}/12 \simeq \mathbb{Z}/4\langle\nu\rangle \oplus \mathbb{Z}/3\langle\alpha\rangle$
4	0	0
5	0	0
6	$\mathbb{Z}/2\left\langle\nu^2\right\rangle$	0

Non-realizable example

Look at stem $k = 3$.

Proposition

Let $n \geq 5$. The (stable) Π-algebra concentrated in degrees $n, n+3$ given by $\pi_n = \mathbb{Z}$ and $\pi_{n+3} = \mathbb{Z}/4$ with structure map

$$\eta_3 \colon \pi_n \otimes_{\mathbb{Z}} Q_3^S \cong \mathbb{Z}/4 \langle \nu \rangle \oplus \mathbb{Z}/3 \langle \alpha \rangle \twoheadrightarrow \mathbb{Z}/4$$

sending ν to 1 is not realizable.

Proof.

$H\mathbb{Z}_4 H\mathbb{Z} \simeq \mathbb{Z}/6$

$\gamma \colon Q_3^S \simeq \mathbb{Z}/4 \langle \nu \rangle \oplus \mathbb{Z}/3 \langle \alpha \rangle \to H\mathbb{Z}_4 H\mathbb{Z}$ sends 2ν to 0. □

Infinite families

Infinite families of indecomposables in π_*^S?

Good candidates: Greek letter elements (suitable representatives in π_*^S).

Fix a prime p.

Infinite families

Proposition

Assume $p \geq 3$.

1. *The first alpha element $\alpha_1 \in Q^S_{2(p-1)-1}$ is **not** in the kernel of γ.*
2. *Higher alpha elements $\alpha_i \in Q^S_{2i(p-1)-1}$ for $i > 1$ are in the kernel of γ.*
3. *Generalized alpha elements $\alpha_{i/j} \in Q^S_*$ for $j > 1$ satisfy $p\alpha_{i/j} \neq 0$ but $\gamma(p\alpha_{i/j}) = 0$.*

Proof.

(3) $\alpha_{i/j}$ has order p^j in π^S_*.
The p-torsion in $H\mathbb{Z}_* H\mathbb{Z}$ is all of order p (and not p^2, p^3, etc.). □

Infinite families

Proposition

Assume $p \geq 5$.
Beta elements $\beta_i \in Q^S_{2(ip^2-i-p)}$ for all $i \geq 1$ are in the kernel of γ.

Upshot

We have found infinite families of non-realizable 2-stage (stable) Π-algebras.

Thank you!

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