Moduli spaces of 2-stage Postnikov systems

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Outline

1. Background
   - Π-algebras and Realizations
   - Classification
   - Obstruction Theory

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2. Truncated Π-algebras
   - Truncation Isomorphism
   - Classification Result
3. Stable 2-types
   - Connected Cover Isomorphism
   - Classification Result

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Definition

- Π $:=$ full subcategory of the homotopy category of pointed spaces consisting of finite wedges of spheres $\bigvee S^{n_i}$, $n_i \geq 1$.
- **Π-algebra** $:=$ product-preserving functor $A \colon \Pi^{\mathrm{op}} \to \mathbf{Set}_*$.

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Example

$\pi_* X = [-, X]_*$ for a pointed space X.

Notation: Write $A_n := A(S^n)$.

Realizations

Realization Problem

Given a Π-algebra A, is there a space X satisfying $\pi_* X \simeq A$ as Π-algebras?

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Classification Problem

If A is realizable, can we **classify** all realizations?

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- Better: **Moduli space** $\mathcal{T}\mathcal{M}(A)$ of realizations.

Remark

Relative moduli space $\mathcal{T}\mathcal{M}'(A)$: Realizations X with identification $\pi_* X \simeq A$.

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- Naive: List of realizations $= \pi_0 \mathcal{TM}(A)$.
- Better: **Moduli space** $\mathcal{TM}(A)$ of realizations.

Remark

Relative moduli space $\mathcal{TM}'(A)$: Realizations X with identification $\pi_* X \simeq A$. Have fiber sequence:

$$\mathcal{TM}'(A) \xrightarrow{\text{forget}} \mathcal{TM}(A) \to B\,\mathrm{Aut}(A)$$

and $\mathcal{TM}(A) \simeq \mathcal{TM}'(A)_{h\,\mathrm{Aut}(A)}$.

Moduli Space

$\mathcal{TM}(A)$ = nerve of the category with

- Objects: Realizations X
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$$\mathcal{TM}(A) \simeq \coprod_{\langle X \rangle} B\,\mathrm{Aut}^h(X).$$

Building $\mathcal{TM}(A)$

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$$
\begin{array}{ccc}
 & & \mathcal{TM} \\
 & \overset{\sim}{\nearrow} & \\
\mathcal{TM}_\infty & \underset{\sim}{\longrightarrow} & \mathrm{holim}_n \, \mathcal{TM}_n \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
 & & \mathcal{TM}_1 \\
 & & \downarrow \\
 & & \mathcal{TM}_0
\end{array}
$$

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- For Y in $\mathcal{TM}_{n-1}$ and $\mathcal{M}(Y) \subseteq \mathcal{TM}_{n-1}$ its component, have:

$$\mathcal{H}^{n+1}(A;\Omega^n A) \to \mathcal{TM}_n(A)_Y \to \mathcal{M}(Y)$$

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- Obstruction to lifting $\in \mathrm{HQ}^{n+2}(A;\Omega^n A)$
- Lifts classified by $\pi_0(\text{fiber}) = \mathrm{HQ}^{n+1}(A;\Omega^n A)$.

Goal

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Describe $\mathcal{T}\mathcal{M}(A)$ in simple cases.

Problem

Can we compute the obstruction groups?

Outline

Truncated Π-algebras

Definition

A Π-algebra A is **n-truncated** if it satisfies $A(S^i) = *$ for all $i > n$.

- Postnikov truncation $P_n : \mathbf{\Pi Alg} \to \mathbf{\Pi Alg}_1^n$

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- Postnikov truncation $P_n : \mathbf{\Pi Alg} \to \mathbf{\Pi Alg}_1^n$
- P_n is left adjoint to inclusion $\iota : \mathbf{\Pi Alg}_1^n \to \mathbf{\Pi Alg}$
- Unit map $\eta_A : A \to P_n A$

Truncation Isomorphism

Theorem (F.)

Let A be a Π-algebra and N a module over A which is n-truncated. Then the natural comparison map

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}_1^n}(P_n A; N) \xrightarrow{\cong} \mathrm{HQ}^*_{\mathbf{\Pi Alg}}(A; N).$$

induced by the Postnikov truncation functor P_n is an iso.

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Proof.

The (simplicially prolonged) left Quillen functor $P_n : s\mathbf{\Pi Alg} \to s\mathbf{\Pi Alg}_1^n$ preserves *all* weak equivalences, and thus cofibrant replacements. □

2-stage Example

- Take $A_i = 0$ for $i \neq 1, n$.

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Theorem

$$\mathcal{TM}(A) \simeq \mathrm{Map}_{BA_1}\left(BA_1, BA_1(A_n, n+1)\right)_{h\,\mathrm{Aut}(A)}$$

Upshot

Classification by a k-invariant is promoted to a **moduli** statement: The **moduli space** of realizations is the **mapping space** where the k-invariant lives.

Corollary

- $\pi_0 \mathcal{T}\mathcal{M}(A) \simeq \mathrm{H}^{n+1}(A_1; A_n) / \operatorname{Aut}(A)$

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- *For any choice of basepoint in $\mathcal{T}\mathcal{M}(A)$, we have:*

$$\pi_i \mathcal{T}\mathcal{M}(A) \simeq \begin{cases} 0, \; i > n \\ \operatorname{Der}(A_1, A_n), \; i = n \\ \mathrm{H}^{n+1-i}(A_1; A_n), \; 2 \leq i < n \end{cases}$$

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and $\pi_1 \mathcal{T}\mathcal{M}(A)$ *is an extension by* $\mathrm{H}^n(A_1; A_n)$ *of a subgroup of* $\operatorname{Aut}(A)$ *corresponding to realizable automorphisms.*

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Highly connected Π-algebras

Definition

A Π-algebra A is **n-connected** if it satisfies $A(S^i) = *$ for all $i \leq n$.

- n-connected cover $C_n : \mathbf{\Pi Alg} \to \mathbf{\Pi Alg}^{\infty}_{n+1}$

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- n-connected cover $C_n \colon \mathbf{\Pi Alg} \to \mathbf{\Pi Alg}_{n+1}^{\infty}$
- C_n is **right** adjoint to inclusion $\iota \colon \mathbf{\Pi Alg}_{n+1}^{\infty} \to \mathbf{\Pi Alg}$
- Counit map $\epsilon_A \colon C_n A \to A$

Connected Cover Isomorphism

Theorem (F.)

Let B be an n-connected Π-algebra and M a module over ιB. Then the natural comparison map

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}}(\iota B; M) \xrightarrow{\cong} \mathrm{HQ}^*_{\mathbf{\Pi Alg}^{\infty}_{n+1}}(B; C_n M)$$

induced by connected cover functor C_n is an iso.

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Proof.

The left Quillen functor $\iota\colon s\mathbf{\Pi Alg}^\infty_{n+1} \to s\mathbf{\Pi Alg}$ preserves *all* weak equivalences. □

Stable 2-types

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Theorem

$\mathcal{TM}'(A)$ is connected and its homotopy groups are:

$$\pi_i \mathcal{TM}'(A) \simeq \begin{cases} 0, \; i \geq 3 \\ \mathrm{Hom}_{\mathbb{Z}}(A_n, A_{n+1}), \; i = 2 \\ \mathrm{Ext}_{\mathbb{Z}}(A_n, A_{n+1}), \; i = 1. \end{cases}$$

Stable 2-types

Corollary

$\mathcal{T}\mathcal{M}(A) \simeq \mathcal{T}\mathcal{M}'(A)_{h\,\mathrm{Aut}(A)}$ is connected; its homotopy groups are:

$$\pi_i \mathcal{T}\mathcal{M}(A) \simeq \begin{cases} 0, \ i \geq 3 \\ \mathrm{Hom}_{\mathbb{Z}}(A_n, A_{n+1}) \ i = 2 \end{cases}$$

and $\pi_1 \mathcal{T}\mathcal{M}(A)$ is an extension of $\mathrm{Aut}(A)$ by $\mathrm{Ext}_{\mathbb{Z}}(A_n, A_{n+1})$. In particular, all automorphisms of A are realizable.

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Remark

Few higher automorphisms.

Further Questions

- Extend to other cases.
- Other realization problems.
- Tools to compute Quillen cohomology.
- Use of algebraic models.

Thank you!

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Reference

Frankland, M. Moduli spaces of 2-stage Postnikov systems. *Topology and its Applications* 158 (2011), no. 11, 1296-1306.