COHOMOLOGY OF THE HAWAIIAN EARRING

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ABSTRACT. In this expository note, we survey some facts about the homology and cohomology of the Hawaiian earring.

Let $E = \bigcup_{n \in \mathbb{N}} C_n$ denote the Hawaiian earring, where $C_n \subset \mathbb{R}^2$ denotes the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$, as in Figure 1. The space $E \subset \mathbb{R}^2$ is endowed with the subspace topology, with the origin $(0, 0) \in E$ as basepoint. The fundamental group $\pi_1(E)$, homology $H_*(E)$, and cohomology $H^*(E)$ have been studied in detail, c.f. [7] [4] [5] [3].

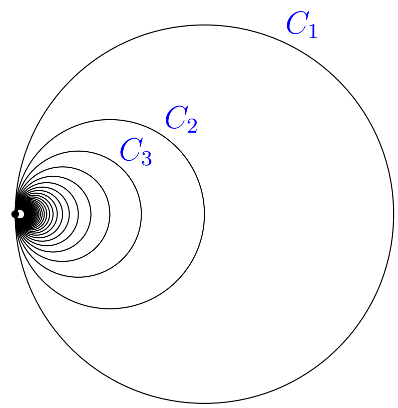

FIGURE 1. The Hawaiian earring E.

In Section 1, we review the computation of the cohomology $H^1(E)$, following loosely the exposition in [1] and [2]. In Section 2, we review the computation of the homology $H_1(E)$ and check directly that dualizing $H_1(E)$ yields $H^1(E)$.

Notation 0.1. Throughout, homology and cohomology will be taken with integer coefficients: $H_*(X) := H_*(X; \mathbb{Z})$.

1. THE SALIENT FACTS

Recall the following from [6, Example 1.25]. Let $r_n : E \to C_n \cong S^1$ denote the retraction which collapses everything outside C_n to the basepoint. These maps together define a pointed map $r : E \to \prod_{n \in \mathbb{N}} S^1$.

Lemma 1.1. *The induced homomorphism on fundamental groups*

$$\pi_1(E) \xrightarrow{\pi_1(r)} \pi_1(\prod_{n \in \mathbb{N}} S^1) = \prod_{n \in \mathbb{N}} \mathbb{Z}$$

is surjective.

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The inclusion maps $\iota_n \colon S^1 \cong C_n \hookrightarrow E$ together define a pointed map $\iota \colon \bigvee_{n\in\mathbb{N}} S^1 \to E$. The composite

$$r\iota \colon \bigvee_{n\in\mathbb{N}} S^1 \to \prod_{n\in\mathbb{N}} S^1$$

is the canonical inclusion. These maps induce on fundamental groups and homology the commutative diagram

$$\begin{array}{ccccccc}
\ast_{n\in\mathbb{N}} \mathbb{Z} = \pi_1(\bigvee_{n\in\mathbb{N}} S^1) & \xrightarrow{\pi_1(\iota)} & \pi_1(E) & \xrightarrow{\pi_1(r)} & \pi_1(\prod_{n\in\mathbb{N}} S^1) = \prod_{n\in\mathbb{N}} \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \cong \\
\bigoplus_{n\in\mathbb{N}} \mathbb{Z} = H_1(\bigvee_{n\in\mathbb{N}} S^1) & \xrightarrow{H_1(\iota)} & H_1(E) & \xrightarrow{H_1(r)} & H_1(\prod_{n\in\mathbb{N}} S^1) = \prod_{n\in\mathbb{N}} \mathbb{Z}.
\end{array}$$

The downward maps $\pi_1(X) \twoheadrightarrow H_1(X) \cong \pi_1(X)_{\mathrm{ab}}$ are the Hurewicz morphism, which is the abelianization.

Before describing the cohomology of the earring, let us recall an algebraic fact proved by Specker [8, Satz III].

Theorem 1.2 (Specker's theorem)**.** *The abelian group* $\mathrm{Hom}_{\mathbb{Z}}(\prod_{n\in\mathbb{N}} \mathbb{Z}, \mathbb{Z})$ *is free on countably many generators, namely the projections* $p_m \colon \prod_{n\in\mathbb{N}} \mathbb{Z} \to \mathbb{Z}$*.*

In other words, the restriction map

$$\mathrm{Hom}_{\mathbb{Z}}(\textstyle\prod_{n\in\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \cong \prod_{n\in\mathbb{N}} \mathbb{Z}$$

is injective, with image $\bigoplus_{n\in\mathbb{N}} \mathbb{Z} \subset \prod_{n\in\mathbb{N}} \mathbb{Z}$. In yet other words, the natural map from $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ to its double dual

$$\textstyle\bigoplus_{n\in\mathbb{N}} \mathbb{Z} \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\prod_{n\in\mathbb{N}} \mathbb{Z}, \mathbb{Z})$$

is an isomorphism.

The following result is found in [4, §3].

Theorem 1.3. *The group homomorphism* $\pi_1(\iota) \colon \pi_1(\bigvee_{n\in\mathbb{N}} S^1) \to \pi_1(E)$ *induces upon dualization a map*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{Gp}}(\pi_1(E), \mathbb{Z}) & \xrightarrow{\pi_1(\iota)^*} & \mathrm{Hom}_{\mathbf{Gp}}(\ast_{n\in\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \\
\| & & \| \\
\mathrm{Hom}_{\mathbb{Z}}(H_1(E), \mathbb{Z}) & \xrightarrow{H_1(\iota)^*} & \mathrm{Hom}_{\mathbb{Z}}(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}, \mathbb{Z}) = \prod_{n\in\mathbb{N}} \mathbb{Z}
\end{array}$$

which is injective, with image $\bigoplus_{n\in\mathbb{N}} \mathbb{Z} \subset \prod_{n\in\mathbb{N}} \mathbb{Z}$*.*

Recall that there is a natural isomorphism $H^1(X) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(H_1(X), \mathbb{Z})$, since $H_0(X)$ is a free abelian group for any space X. In particular, Theorem 1.3 gives the cohomology of the Hawaiian earring

$$H^1(E) \cong \bigoplus_{n\in\mathbb{N}} \mathbb{Z},$$

a free abelian group on countably many generators.

In view of Lemma 1.1 and Specker's theorem, Theorem 1.3 can be reinterpreted as follows.

Theorem 1.4. *The surjective map* $H_1(r)\colon H_1(E) \twoheadrightarrow H_1(\prod_{n\in\mathbb{N}} S^1) = \prod_{n\in N} \mathbb{Z}$ *induces upon dualization an isomorphism*

$$\mathrm{Hom}_{\mathbb{Z}}(\textstyle\prod_{n\in\mathbb{N}} \mathbb{Z}, \mathbb{Z}) \xrightarrow[\cong]{H_1(r)^*} \mathrm{Hom}_{\mathbb{Z}}(H_1(E), \mathbb{Z}).$$

2. Dualizing the homology

The homology $H_1(E)$ has been computed by Eda and Kawamura [5, Theorem 3.1].

Theorem 2.1. *Let* $\mathfrak{c} = 2^{\aleph_0} = |\mathbb{R}|$ *denote the cardinality of the continuum. Then the map* $H_1(r)\colon H_1(E) \twoheadrightarrow H_1(\prod_{n\in\mathbb{N}} S^1)$ *is a split epimorphism with kernel*

$$\prod_{p \text{ a prime}} \Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge}\Big)_p^{\wedge} \oplus \bigoplus_{\mathfrak{c}} \mathbb{Q},$$

where $M_p^{\wedge} = \lim_i M/p^i M$ *denotes the* p*-adic completion of an abelian group* M*, and in particular* $\mathbb{Z}_p^{\wedge}$ *denotes the* p*-adic integers. Consequently, there is a direct sum decomposition*

$$H_1(E) \cong \prod_{n\in\mathbb{N}} \mathbb{Z} \oplus \prod_{p \text{ a prime}} \Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge}\Big)_p^{\wedge} \oplus \bigoplus_{\mathfrak{c}} \mathbb{Q}.$$

Remark 2.2. There is a typo in the MathSciNet review of [5], missing the outer completion in $(\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge})_p^{\wedge}$. The direct sum $\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge} \subset \prod_{\mathfrak{c}} \mathbb{Z}_p^{\wedge}$ consists of the finitely supported families. Its p-completion $(\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge})_p^{\wedge} \subset \prod_{\mathfrak{c}} \mathbb{Z}_p^{\wedge}$ consists of the p-adically decaying families, i.e., families $(x_t)_{t\in\mathfrak{c}} \in \prod_{\mathfrak{c}} \mathbb{Z}_p^{\wedge}$ such that for every $i \in \mathbb{N}$, the family of cosets

$$(x_t + p^i \mathbb{Z}_p^{\wedge}) \in \prod_{\mathfrak{c}} \mathbb{Z}/p^i\mathbb{Z}$$

is finitely supported.

As a reality check, we start from Theorem 2.1 to provide an alternate proof of Theorem 1.4.

Alternate proof. Given the homology result (Theorem 2.1), the cohomology result (Theorem 1.4) is equivalent to the statement

$$\begin{aligned} 0 &= \mathrm{Hom}_{\mathbb{Z}}\left(\prod_p \Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge}\Big)_p^{\wedge} \oplus \bigoplus_{\mathfrak{c}} \mathbb{Q}, \mathbb{Z}\right) \\ &= \mathrm{Hom}_{\mathbb{Z}}\left(\prod_p \Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge}\Big)_p^{\wedge}, \mathbb{Z}\right) \times \mathrm{Hom}_{\mathbb{Z}}\Big(\bigoplus_{\mathfrak{c}} \mathbb{Q}, \mathbb{Z}\Big). \end{aligned}$$

The equality $\mathrm{Hom}_{\mathbb{Z}}(\bigoplus_{\mathfrak{c}} \mathbb{Q}, \mathbb{Z}) = 0$ holds, since $\bigoplus_{\mathfrak{c}} \mathbb{Q}$ is a divisible abelian group. The equality

$$\mathrm{Hom}_{\mathbb{Z}}\left(\prod_p \Big(\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge}\Big)_p^{\wedge}, \mathbb{Z}\right) = 0$$

holds by Lemma 2.3, since each $(\bigoplus_{\mathfrak{c}} \mathbb{Z}_p^{\wedge})_p^{\wedge}$ is p-complete, in particular p-local. □

Lemma 2.3. *Let* $\{M_p\}_p$ *be a family of abelian groups indexed by the prime numbers* p*, where each* M_p *is* p*-local. Then the only* $\mathbb{Z}$*-linear map* $\prod_p M_p \to \mathbb{Z}$ *is the zero map.*

Proof. Let $\varphi\colon \prod_p M_p \to \mathbb{Z}$ be a $\mathbb{Z}$-linear map. Let $x = (x_p) \in \prod_p M_p$ be a family with at least one zero component, say the component $x_q = 0$. Then x is a q-divisible element, since each abelian group M_p with $p \neq q$ is q-divisible. Hence, $\varphi(x) \in \mathbb{Z}$ is q-divisible, which forces $\varphi(x) = 0$.

Now let $x \in \prod_p M_p$ be arbitrary. Writing

$$\begin{aligned} x &= (x_2, x_3, x_5, \ldots) \\ &= (x_2, 0, 0, \ldots) + (0, x_3, x_5, \ldots) \\ &= x_2 + (x - x_2), \end{aligned}$$

we obtain

$$\varphi(x) = \varphi(x_2) + \varphi(x - x_2) = 0$$

since both families x_2 and $x - x_2$ have at least one zero component. □

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