Completed power operations for Morava E-theory

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Outline

Operations

$\boxed{\text{Topology}} \rightsquigarrow \boxed{\text{Algebra}}$

Let R be a ring spectrum and X a spectrum.
$R^* X$ is an $R^* R$-module.
$R_* X$ is an $R_* R$-comodule.

Power operations

Let K be complex periodic K-theory and X a space. Exterior powers of vector bundles induce operations

$$\lambda^k \colon K^0(X) \to K^0(X)$$

making $K^0(X)$ into a λ**-ring.**
Obtain Adams operations $\psi^k \colon K^0(X) \to K^0(X)$.
Operation $\theta^p \colon K^0(X) \to K^0(X)$ satisfying

$$\psi^p(x) = x^p + p\theta^p(x).$$

Every λ-ring has an underlying θ**-ring.**

Power operations (cont'd)

For X a spectrum, the homology

$$H_*(\Omega^\infty X; \mathbb{F}_p)$$

supports *Dyer-Lashof operations*.
More generally, if A is a commutative $H\mathbb{F}_p$-algebra, then its homotopy $\pi_* A$ is an algebra over the Dyer-Lashof algebra.

Commutative algebras

Free commutative algebra monad

$$\mathbb{P}\colon \mathrm{Mod}_R \to \mathrm{Mod}_R.$$

Induces

$$\mathbb{P}\colon h\mathrm{Mod}_R \to h\mathrm{Mod}_R$$

which is the free $\mathbb{H}_\infty$ R-algebra monad.
$\mathbb{P}M = \bigvee_{n\geq 0} \mathbb{P}_n M$ where

$$\mathbb{P}_n M = (\overbrace{M \wedge_R \ldots \wedge_R M}^{n \text{ times}})_{h\Sigma_n}$$

is the n^{th} *extended power* of M.

Commutative algebras (cont'd)

Question

What algebraic structure is present on the homotopy of commutative R-algebras?

$$\pi_* : \mathrm{Alg}_R \to \boxed{?}$$

$\boxed{?} = \mathrm{Mod}_{R_*} +$ *extra structure...*

Morava E-theory

FROM NOW ON

Fix a prime p and height $h \geq 1$. Take $R = E = E_h$, Morava E-theory at chromatic height h.

$$E_* = W\mathbb{F}_{p^h}[\![u_1, \ldots, u_{h-1}]\!][u^{\pm}]$$

with $|u_i| = 0$ and $|u| = 2$, and W means (p-typical) Witt vectors.
E_* is a complete Noetherian regular local (graded) ring with maximal ideal $\mathfrak{m} = (p, u_1, \ldots, u_{h-1}) \subset E_*$

Work $K(h)$-locally

Write $L_K := L_{K(h)}$ for Bousfield localization with respect to Morava K-theory $K(h)$.
Power operations for Morava E-theory have been studied (Ando-Hopkins-Strickland, Rezk, ...) and are understood in the $K(h)$-local category, because of the following.

Theorem (Rezk, using work of Strickland)

If $F \simeq \bigvee_{i=1}^{k} \Sigma^{d_i} E$ is a finitely generated free E-module, then so is $L_K \mathbb{P}_n F$.

Algebraic approximation functors

[Rezk 2009] Functors $\mathbb{T}_n \colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ defined as left Kan extension of the functor defined by

$$\mathbb{T}_n(\pi_* F) = \pi_* L_K \mathbb{P}_n F$$

for F finitely generated free. In particular:

$$\mathbb{T}_n(E_*) = E_*^\wedge(B\Sigma_{n+})$$

where $E_*^\wedge(X) := \pi_* L_K(E \wedge X)$ is the **completed E-homology** of X.
Note: $\mathbb{T}_n$ preserves filtered colimits and reflexive coequalizers.

Algebraic approximation functors (cont'd)

$$\mathbb{T} = \bigoplus_{n \geq 0} \mathbb{T}_n \colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$$

inherits a monad structure from that of $\mathbb{P}$, or rather $\hat{\mathbb{P}} := L_K \mathbb{P}$. Here, we use the equivalence $L_K \mathbb{P} \xrightarrow{\simeq} L_K \mathbb{P} L_K$:

$$\begin{array}{ccc} (L_K\mathbb{P})(L_K\mathbb{P}) & \overset{\hat{\mu}}{\dashrightarrow} & L_K\mathbb{P} \\ \simeq \uparrow & \nearrow_{L_K\mu} & \\ L_K\mathbb{P}\mathbb{P} & & \end{array}$$

If A is a $K(h)$-local commutative E-algebra, then $\pi_* A$ is naturally a $\mathbb{T}$-algebra.

$$\pi_* L_K \colon \mathrm{Alg}_E \to \mathrm{Alg}_{\mathbb{T}}$$

L-complete modules

$\mathfrak{m}$-adic completion

$$(-)^{\wedge}_{\mathfrak{m}} \colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$$

is neither left nor right exact. Let $L_s \colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ be its s^{th} left derived functor. In particular:

$$M \xrightarrow{\eta} L_0 M \to M^{\wedge}_{\mathfrak{m}}$$

and M is ***L*-complete** if η is an isomorphism. Note that E_* is L-complete.

Let $\widehat{\mathrm{Mod}}_{E_*} \subset \mathrm{Mod}_{E_*}$ denote the full subcategory of L-complete modules, which is a reflective abelian subcategory, with reflector $L_0 \colon \mathrm{Mod}_{E_*} \to \widehat{\mathrm{Mod}}_{E_*}$.

L-complete modules (cont'd)

Proposition

An E-module M is $K(h)$-local if and only if its homotopy $\pi_* M$ is an L-complete E_*-module.

Therefore, $L_0 \mathbb{T}$ better approximates the homotopy of $K(h)$-local commutative E-algebras.

Completed algebraic approximation functors

Let $\widehat{\mathrm{Alg}}_E \subset \mathrm{Alg}_E$ denote the full subcategory of $K(h)$-local commutative E-algebras.

$$\begin{array}{ccc}
\mathrm{Alg}_E & & \mathrm{Mod}_{E_*} \circlearrowleft \mathbb{T} \\
L_K \downarrow \uparrow & \overset{\pi_*}{\nearrow} & L_0 \downarrow \uparrow \iota \\
\widehat{\mathrm{Alg}}_E & \xrightarrow{\pi_*} & \widehat{\mathrm{Mod}}_{E_*} \circlearrowleft \hat{\mathbb{T}}
\end{array}$$

The problem

Question

Is $\hat{\mathbb{T}} = L_0\mathbb{T}\iota \colon \widehat{\mathrm{Mod}}_{E_*} \to \widehat{\mathrm{Mod}}_{E_*}$ *also a monad?*

If $\mathbb{T}$ preserves L_0-equivalences, i.e., $L_0\mathbb{T} \xrightarrow{\simeq} L_0\mathbb{T}L_0$, then $\hat{\mathbb{T}}$ inherits a monad structure from $\mathbb{T}$.

$$\begin{array}{ccc} (L_0\mathbb{T})(L_0\mathbb{T}) & \overset{\hat{\mu}}{\dashrightarrow} & L_0\mathbb{T} \\ \uparrow{\scriptstyle \simeq} & \nearrow{\scriptstyle L_0\mu} & \\ L_0\mathbb{T}\mathbb{T} & & \end{array}$$

Main result

Theorem (Barthel-F.)

The algebraic approximation functor $\mathbb{T}\colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ *preserves* L_0*-equivalences, i.e., the natural map*

$$L_0\mathbb{T}(M) \xrightarrow{L_0\mathbb{T}\eta} L_0\mathbb{T}L_0(M)$$

is an isomorphism for all E_**-module* M.

Corollary

The completed algebraic approximation functor $\hat{\mathbb{T}} = L_0\mathbb{T}\iota\colon \widehat{\mathrm{Mod}}_{E_*} \to \widehat{\mathrm{Mod}}_{E_*}$ *inherits a monad structure from that of* $\mathbb{T}$.

Upshot

The homotopy of $K(h)$-local commutative E-algebras takes values in $\hat{\mathbb{T}}$-algebras:

$$\pi_* \colon \widehat{\mathrm{Alg}}_E \to \widehat{\mathrm{Alg}}_{\mathbb{T}} \cong \mathrm{Alg}_{\hat{\mathbb{T}}}.$$

Upshot: L-completeness is now built into the algebraic structure.

Applications

- Computations with power operations at higher height [Rezk].
- Compute $\pi_*(\hat{\mathbb{P}}M)$ for any E-module M. The comparison map

  $$\hat{\mathbb{T}}(\pi_* M) \to \pi_*(\hat{\mathbb{P}}M)$$

  is an isomorphism when M is flat.
- Compute topological André-Quillen homology $TAQ(X)$ of a $K(h)$-local commutative E-algebra [Behrens-Rezk].

Height 1 case

Consider height $h = 1$.

- $E = K_p^\wedge$ is p-complete K-theory.
- $E_* = \mathbb{Z}_p[u^\pm]$ with maximal ideal $\mathfrak{m} = (p) \subset E_*$.
- L_0 is "Ext-p-completion"

$$L_0 M = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$$

or, equivalently, "analytic p-completion"

$$L_0 M = M[\![x]\!]/(x-p)M[\![x]\!].$$

Algebraic approximation functor at height 1

By work of McClure and Bousfield:

Theorem

At height $h = 1$, the monad $\mathbb{T} \colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ is the free $\mathbb{Z}/2$-graded θ-ring over $\mathbb{Z}_p$.

Main result at height 1

The main result (that $\mathbb{T}$ preserves L_0-equivalences) can be proved more explicitly at height 1.

- Alternate proof based on formulas and the combinatorics of λ-rings.
- Alternate proof based on the representation theory of symmetric groups.

Thank you!