# RESEARCH STATEMENT

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## Background: Operations and realization problems

My research interests lie in algebraic topology and homotopy theory, with a focus on operations. In particular, I have worked on homotopy operations, higher cohomology operations, power operations in chromatic homotopy theory, and cohomology operations in motivic homotopy theory.

One of the main ideas of algebraic topology is to describe spaces by associating algebraic invariants to them, such as homotopy groups $\pi_* X$ and cohomology groups $H^* X$. The cohomology $H^* X$ is more than a graded ring, as it comes equipped with an action of *cohomology operations*, i.e., natural transformations $H^n X \to H^{n+k} X$. For example, the mod $p$ cohomology $H^*(X; \mathbb{F}_p)$ of a space $X$ is an unstable algebra over the Steenrod algebra, the algebra of (stable) cohomology operations. J.F. Adams used the Steenrod algebra in an essential way in the Adams spectral sequence. To this day, the Adams spectral sequence (along with its generalizations) remains one of the most powerful tools for computations in stable homotopy theory, in particular for computing stable homotopy groups of spheres. Broadly speaking, my research addresses the following.

**Question.** *How much information about a space is remembered by its algebraic invariants, taking into account the operations?*

The classic Steenrod problem asks which unstable algebras over the Steenrod algebra can be realized as the mod $p$ cohomology of a space. This is an example of *realization problem*.

Table 1 provides an overview of my research, listed by theme and by timeline. Here is a summary of each theme, including some future directions I want to investigate. More details are provided afterwards in Sections 1 through 5.

**1. Homotopy operations.** My earlier work was focused on the realization of $\Pi$-algebras, an algebraic structure encoding operations on the homotopy groups of a space. Given a space $X$, one has an associated $\Pi$-algebra $\pi_* X$, and the realization problem asks the converse: Given a $\Pi$-algebra $A$, does there exist a space which gives rise to it? Each such choice of space is called a realization of $A$. Using obstruction theory, I described the moduli space of realizations of certain $\Pi$-algebras concentrated in two degrees. In joint work with H.J. Baues, we provided necessary and sufficient conditions for the realizability of a $\Pi$-algebra concentrated in two degrees. We also provided infinite families of $\Pi$-algebras that are not realizable. In ongoing work with D. Isaksen, we are relating obstructions to realizability to higher homotopy operations.

**2. Homotopical and homological algebra.** The aforementioned obstruction theory has obstruction classes in Quillen cohomology. Motivated by this, I studied some properties of Quillen cohomology via simplicial model categories, and computed some explicit obstruction groups.

Triangulated categories are another useful tool to encode operations and obstructions. With D. Christensen, we showed that the Adams differentials in a triangulated category are given by specific higher Toda brackets, and described applications to computing maps between module spectra

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over a ring spectrum. In current work with K. Szumiło, we are studying properties of the underlying *derivator* of a model category, which remembers more information than a triangulated category.

Future goal: Prove the Moss convergence theorem in a general triangulated category. This theorem allows the computation of Toda brackets with the Adams spectral sequence.

**3. Higher comohology operations.** In ongoing work with H.J. Baues, we are studying the algebra of higher order operations in mod $p$ cohomology. The main goal is to compute the differential $d_3$ in the Adams spectral sequence, using tertiary operations. We have described a certain algebraic structure that determines $d_3$. We have shown that higher cohomology operations satisfy higher distributivity laws.

Future goals: Prove a strictification theorem for higher cohomology operations. Compute the Adams differential $d_2$ via secondary cohomology operations.

**4. Chromatic homotopy theory.** I have also worked on power operations in chromatic homotopy theory. At chromatic height 1, these operations are related to $\lambda$-rings. Examples of $\lambda$-rings include the topological $K$-theory $K^0(X)$ of a space, with the $\lambda$-structure induced by exterior powers of vector bundles, or the representation ring $R(G)$ of a group, with the $\lambda$-structure induced by exterior powers of representations. C. Rezk provided a construction that encodes the algebraic structure found in the homotopy of a $K(n)$-local algebra over Morava $E$-theory at chromatic height $n$. In joint work with T. Barthel, we improved the construction to better deal with completions (Theorem 4.1).

Future goals: Use Theorem 4.1 to calculate power operations for Morava $E$-theory and topological André–Quillen homology.

**5. Motivic homotopy theory.** More recently, I started working on cohomology operations in motivic homotopy theory, a branch of homotopy theory applied to algebraic geometry. In motivic homotopy theory, one can associate algebraic invariants not only to spaces, but also to smooth schemes. Together with M. Spitzweck, we are working on a conjecture about the mod $p$ dual motivic Steenrod algebra over a base field of characteristic $p$. The known theorem for mod $\ell$ cohomology with $\ell \neq p$ has been used successfully to compute the slice spectral sequence, which relates motivic cohomology to algebraic $K$-theory [Spi12] [RØ16].

TABLE 1. Overview of my research

| Theme | Done | Short term | Mid term | Long term |
|---|---|---|---|---|
| **1. Homotopy operations** | 1.1. Moduli spaces<br>1.2. Criterion for realizability | | 1.3. Realization of $\pi_*$-modules | |
| **2. Homotopical and homological algebra** | 2.1. Quillen cohomology<br>2.2. Higher Toda brackets | | 2.3. Direct Reedy diagrams<br>2.4. Simplicial modules | 2.5. Relative derived categories |
| **3. Higher cohomology operations** | 3.1. Two-track algebras<br>3.2. Higher distributivity | 3.3. DG-category of secondary operations | | 3.4. Strictification of near-rings<br>3.5. Adams $d_2$ for im $J$ |
| **4. Chromatic homotopy theory** | 4.1. Completed power operations | | | 4.2. Calculations of power operations<br>4.3. Cohomology of $\widehat{\mathbb{T}}$-algebras |
| **5. Motivic homotopy theory** | Some retracts | 5.1. Motivic Steenrod algebra | | |

## 1. Homotopy operations and $\Pi$-algebras

The homotopy groups $\pi_i X$ of a pointed space $X$ are an important algebraic invariant of spaces and have been studied extensively. They are a collection of groups (abelian for $i > 1$) that carry additional structure: a $\pi_1$-action on higher groups, Whitehead products $\pi_i \times \pi_j \to \pi_{i+j-1}$, and precomposition operations $\pi_j(S^n) \times \pi_n \to \pi_j$ induced by maps between spheres. This algebraic structure is known as a $\Pi$-**algebra**. The prototypical $\Pi$-algebra is the homotopy $\Pi$-algebra $\pi_* X$ of a pointed space $X$, which defines the functor $\pi_* \colon \mathbf{Top}_* \to \mathbf{\Pi Alg}$.

**Problem.** *Given a $\Pi$-algebra $A$, is there a space $X$ satisfying $A \simeq \pi_* X$ as $\Pi$-algebras? In other words, can $A$ be topologically realized, and if so, can we classify all realizations up to weak homotopy equivalence?*

The simplest kind of $\Pi$-algebra is one concentrated in one degree $n$; such a $\Pi$-algebra is realizable by an Eilenberg–MacLane space $K(A_n, n)$, uniquely up to weak equivalence. However, the analogous statement becomes false if there are two non-trivial homotopy groups $A_n$ and $A_{n+k}$ with prescribed homotopy operations between them.

In [BDG04], the *moduli space* $\mathcal{TM}(A)$ of all realizations of a $\Pi$-algebra $A$ is built as the limit of a tower whose layers are controlled by Quillen cohomology groups of $A$. A similar Dwyer–Kan–Stover obstruction theory had been successfully applied to problems about $E_\infty$ ring spectra [GH04].

The moduli space $\mathcal{TM}(A)$ provides an improved classification. The set of path components $\pi_0 \mathcal{TM}(A)$ recovers the usual classification: it is the set of all realizations of $A$ (weak homotopy types). Then $\pi_1 \mathcal{TM}(A)$ based at a realization $X$ corresponds to automorphisms of $X$, $\pi_2 \mathcal{TM}(A)$ corresponds to automorphisms of automorphisms, and so on.

### 1.1. Moduli spaces of realizations.

Using the obstruction theory of [BDG04], along with computations of obstruction groups, I obtained *classification* results for some realizable $\Pi$-algebras concentrated in two degrees [Fra11, Theorems 3.4 and 5.1]. More precisely, I provided a complete description of the moduli space of realizations $\mathcal{TM}(A)$ for $\Pi$-algebras $A$ of the following form:

- The non simply-connected case: $A$ is concentrated in degrees 1 and $n$ for some $n \geq 2$.
- The connected stable 2-types: $A$ is concentrated in degrees $n$ and $n+1$ for some $n \geq 2$.

### 1.2. Criterion for realizability.

In [BF15, Theorem 4.2], H.J. Baues and I provided necessary and sufficient conditions for the realizability of a $\Pi$-algebra concentrated in two degrees $n$ and $n+k$. The statement involves the homology of Eilenberg–MacLane spaces, an algebraic structure that is known in principle. Using this criterion, we listed infinite families of such $\Pi$-algebras that are not realizable, coming from known infinite families in stable homotopy groups of spheres [BF15, Proposition 6.7].

### 1.3. Realization of $\pi_*$-modules.

The stable analogue of the realization problem is to realize a $\pi_*$-module as the homotopy $\pi_* X$ of a spectrum $X$, where $\pi_* = \pi_*(S^0)$ denotes the stable homotopy ring. The homotopy groups $\pi_* X$ support higher homotopy operations, in the form of Toda brackets involving classes in $\pi_*$ (primary operations) and a class in $\pi_* X$. Following discussions with D. Isaksen, we want to make precise the statement that these higher homotopy operations are the *only obstructions* to realizability. Consider a $\pi_*$-module concentrated in a range where only secondary operations could appear, i.e., 3-fold Toda brackets. Since the indeterminacy of 3-fold Toda brackets behaves well, we expect a clean statement of the form: A $\pi_*$-module concentrated in that range is realizable if and only if it can be endowed with secondary homotopy operations, to be described explicitly.

## 2. Homotopical and homological algebra

While working on homotopy operations and cohomology operations, one comes across useful tools from homotopical and homological algebra. I am particularly interested in Quillen cohomology, simplicial model categories, and triangulated categories. André–Quillen cohomology was introduced in the 1960s as a tool to solve problems in commutative algebra using methods from homotopy theory. Since then, it has found many applications in topology and in algebra [GS07].

2.1. **Obstruction classes in Quillen cohomology.** I proved that Quillen cohomology $AQ^*_{\mathbf{\Pi Alg}}(A; M)$ of a $\Pi$-algebra $A$ with coefficients in a module $M$ can be computed in the category of $n$-truncated $\Pi$-algebra if $M$ is $n$-truncated, and in the category of $n$-connected $\Pi$-algebras if $A$ is $n$-connected [Fra15, Theorem 5.16] [Fra11, Theorem 4.10]. These results lead to concrete calculations of some obstruction groups. More generally, I described the comparison maps and spectral sequences induced on Quillen (co)homology by an adjunction between two algebraic categories [Fra15, Theorem 4.7].

2.2. **Higher Toda brackets in triangulated categories.** Triangulated categories appear in many contexts in topology and in algebra, for example, the stable homotopy category and the derived category of an abelian category. Given a stable model category $C$, its homotopy category $\mathrm{Ho}(C)$ inherits a triangulated structure, with which one can do a lot. Notably, the Adams spectral sequence is available in any triangulated category equipped with an injective (or projective) class. Higher Toda brackets are also available in any triangulated category.

In [CF17], D. Christensen and I showed that the differential $d_r$ of the Adams spectral sequence in a triangulated category is given by a certain $(r+1)$-fold Toda bracket involving primary cohomology operations, generalizing a theorem of Maunder about the classical Adams spectral sequence [CF17, Theorem 6.5]. We described applications to the universal coefficient spectral sequence

$$\mathrm{Ext}^{*,*}_{\pi_* R}(\pi_* M, \pi_* N) \Rightarrow [M, N]_R$$

computing maps in the homotopy category of $R$-module spectra, for a sufficiently sparse ring spectrum $R$ [CF17, Theorem 7.14].

2.3. **Homotopy theory of direct Reedy diagrams.** The homotopy category $\mathrm{Ho}(C)$ equipped with its triangulated structure does not remember the whole homotopy theory of $C$. There are examples of triangulated categories which admit models that are not Quillen equivalent [DS09]. A result of Renaudin guarantees that for nice enough $C$, the derivator of $C$ determines the homotopy theory of $C$. The **derivator** consists of all the homotopy categories $\mathrm{Ho}(C^I)$ of diagram categories $C^I$ for small categories $I$, together with restriction and extension functors between them.

In current work with K. Szumiło [FS18a], we are investigating how much information about the homotopy categories $\mathrm{Ho}(C^I)$ can be recovered by considering only the case where $I$ is a direct Reedy category. Examples of direct Reedy categories include the natural numbers $\mathbb{N}$ viewed as a poset, or the category $\Delta_{\mathrm{inj}}$ of finite ordinals and injective order-preserving maps. There is a natural functor $DI \to I$ that relates a small category $I$ to a direct Reedy category $DI$. We prove that the induced restriction functor $\mathrm{Ho}(C^I) \to \mathrm{Ho}(C^{DI})$ is fully faithful, and describe some applications. One application extends a result in [DS09] about the two non-equivalent derivators discussed therein. Another application appears in [LNS17].

2.4. **Homotopy theory of simplicial modules.** In my work on Quillen cohomology [Fra15], I studied some properties of simplicial modules over an object $X$ of an algebraic category $C$. Here, a **Beck module** over $X$ is an abelian group object in the slice category over $X$. For some applications, one also needs simplicial modules over *simplicial* objects $X_\bullet \in s\mathcal{C}$. This is conveniently described using the fibered category of Beck modules $\mathrm{Mod}\mathcal{C} \to \mathcal{C}$, also known as tangent category of $\mathcal{C}$, whose fiber over an object $X \in \mathcal{C}$ is the category $(\mathcal{C}/X)_{\mathrm{ab}}$ of Beck modules over $X$. I studied some properties of this construction in [Fra10].

In [Fra18], I show that for nice enough $\mathcal{C}$, the category $s\mathrm{Mod}\mathcal{C}$ of simplicial objects in $\mathrm{Mod}\mathcal{C}$ admits a model structure which induces a model structure on each fiber category, i.e., the category of Beck modules over each simplicial object $X_\bullet \in s\mathcal{C}$. This generalizes the work of Quillen on simplicial modules over simplicial groups or simplicial commutative rings. The model structure on $s\mathrm{Mod}\mathcal{C}$ agrees after the fact with the integral model structure of [HP15].

2.5. **Relative derived categories.** Given a triangulated category $\mathcal{T}$ with a projective class $\mathcal{P}$, the $E_2$ term of the $\mathcal{P}$-relative Adams spectral sequence is given by $\mathcal{P}$-relative Ext groups $\mathrm{Ext}^{*,*}_{\mathcal{P}}(X, Y)$. Following our work in [CF17] exhibiting the differentials as higher Toda brackets in $\mathcal{T}$, we would like to prove the Moss convergence theorem in this level of generality. This theorem, proved by Moss for the classical Adams spectral sequence, relates the Massey product of permanent cycles on the $E_2$ term to the Toda bracket of the maps that the cycles represent. Massey products are an algebraic analogue of Toda brackets, defined in cohomology rather than in homotopy.

A priori, Toda brackets in $\mathcal{T}$ and Massey products of Ext classes are very different things. However, one can form the $\mathcal{P}$-relative derived category $D_{\mathcal{P}}(\mathcal{T})$, a triangulated category such that the groups $\mathrm{Ext}^{*,*}_{\mathcal{P}}(X, Y)$ are particular hom-groups in $D_{\mathcal{P}}(\mathcal{T})$. From this point of view, Massey products of Ext classes are merely Toda brackets in this new triangulated category $D_{\mathcal{P}}(\mathcal{T})$. We want to develop this machinery of relative derived categories in order to prove a generalized Moss convergence theorem, using formal manipulations of Toda brackets in triangulated categories. H. Miller pointed out that in order to construct the Moss pairing of spectral sequences, we may need an enhancement of triangulated categories, such as higher triangulations or stable derivators.

## 3. Higher cohomology operations

The $\Pi$-algebra structure on the homotopy groups $\pi_* X$ encodes primary homotopy operations. Likewise, the structure of the cohomology $H^*(X; \mathbb{F}_p)$ as an unstable algebra over the Steenrod algebra encodes primary cohomology operations. The additional structure of *higher order operations* is also useful. For example, H. Toda used higher order homotopy operations (Toda brackets) to compute several homotopy groups of spheres. Given classes $a, b, c \in \pi_* S$ satisfying $ab = 0, bc = 0$, the Toda bracket $\langle a, b, c \rangle \subseteq \pi_{|a|+|b|+|c|+1} S$ consists of classes built by choosing null-homotopies of $ab$ and $bc$. Also, J.F. Adams used secondary cohomology operations in his classic solution to the Hopf Invariant One problem.

If one takes into account all higher cohomology operations on $H^*(X; \mathbb{F}_p)$, one can recover the $p$-type of the space or spectrum $X$. The $H\mathbb{F}_p$ based Adams spectral sequence is a computational manifestation of that fact. Rather than using the triangulated structure as we do in [CF17], one can study higher order cohomology operations via a topological enrichment. Consider the topologically enriched category $\mathcal{EM}$ consisting of finite products of Eilenberg–Maclane spectra

$$\Sigma^{n_1} H\mathbb{F}_p \times \cdots \times \Sigma^{n_k} H\mathbb{F}_p$$

and mapping spaces between them. This category $\mathcal{EM}$ encodes all higher order operations in mod $p$ cohomology. The homotopy category $\pi_0 \mathcal{EM}$ encodes primary operations, i.e., the Steenrod algebra; $\mathcal{EM}$ can be viewed as the **"topological Steenrod algebra"**. Taking the fundamental groupoid of each mapping space yields a groupoid-enriched category (also known as **track category**) $\Pi_1 \mathcal{EM}$ which encodes secondary operations.

In [Bau06], H.J. Baues proved a strictification result for the algebra of secondary cohomology operations: they can be encoded by a certain 1-truncated differential (bi)graded algebra over $\mathbb{Z}/p^2$. This was used in [BJ11] to algorithmically compute the differential $d_2$ of the Adams spectral sequence. In current work with H.J. Baues, we are pursuing the program one step further, aiming to show that the algebra of tertiary cohomology operations can be encoded by a certain 2-truncated DG-algebra, and using this to compute the differential $d_3$.

3.1. **Two-track algebras and the Adams $d_3$.** In [BB15], H.J. Baues and D. Blanc described an algebro-combinatorial structure which encodes just enough information to compute the differential $d_{n+1}$. In [BF16, Theorem 7.3], we specialized that work to the case $n = 2$ and described a more concrete algebraic structure which encodes enough information to compute the differential $d_3$.

3.2. **Higher distributivity up to homotopy.** One difficulty with higher order cohomology operations is that they do not form an algebra. In the "topological Steenrod algebra" $\mathcal{EM}$, composition is left linear (strictly) and right linear up to coherent homotopy. In [BF17], we introduced a hierarchy of higher distributivity laws to describe this coherence data. A 1-distributor consists of a family of paths in mapping spaces between $a(x+y)$ and $ax+ay$; these paths witness that composition is right linear up to homotopy. An $n$-distributor consists of a family of $n$-cubes in mapping spaces whose extreme corners are $a(x_0 + \ldots + x_n)$ and $ax_0 + \ldots + ax_n$, satisfying certain compatibility conditions. We proved that $\mathcal{EM}$ is $\infty$-distributive, in a more or less canonical way [BF17, Theorem 5.10].

3.3. **The DG-category of secondary operations.** In [BF18], we revisit the strictification theorem for secondary operations from [Bau06]. We provide a streamlined argument which yields a more general result, giving sufficient conditions for a track category to be equivalent to a 1-truncated DG-category. The track category $\Pi_1\mathcal{EM}$ satisfies said conditions; this yields the original strictification theorem.

3.4. **Strictification of near-rings.** Operads have been used to tackle strictification problems for algebraic structure up to homotopy, notably $A_\infty$ operads for higher associativity and $E_\infty$ operads for higher commutativity. However, operads are not well suited to describe the higher distributivity appearing in higher order cohomology operations. Algebraic theories (in the sense of Lawvere, i.e., categories with finite products) are better suited for this purpose. Let $\mathcal{T}_{\text{ring}}$ denote the theory of rings and $\mathcal{T}_{\text{near-ring}}$ the theory of **"near-rings"**, where one drops the right linearity equation $a(x + y) = ax + ay$. Restriction along the map of theories $\mathcal{T}_{\text{near-ring}} \to \mathcal{T}_{\text{ring}}$ induces the forgetful functor that views a ring as a special case of near-ring. Our work in [BF17] suggests that there is a topologically enriched theory $\mathcal{T}_{D_\infty}$ of $\infty$-distributive rings, sitting in a diagram of theories

$$\mathcal{T}_{\text{near-ring}} \longrightarrow \mathcal{T}_{D_\infty} \xrightarrow{\sim} \mathcal{T}_{\text{ring}}$$

where the second step is a Dwyer–Kan equivalence. As observed by N. Wahl [Wah02], this would suffice to prove that every $\infty$-distributive topological ring is equivalent to a (strict) topological ring. This approach will provide a more conceptual explanation for the strictification results of [BF18], obtained by a hands-on approach.

3.5. **Calculations using secondary cohomology operations.** Following an observation of R. Bruner, the connective $\operatorname{Im} J$ spectrum has a manageable $H\mathbb{F}_p$ based Adams spectral sequence. I want to apply the Baues–Jibladze algorithm to calculate the Adams $d_2$ for that spectrum, both to confirm known results and as a test case for the algorithm.

## 4. Chromatic homotopy theory

Chromatic homotopy theory studies spectra by filtering them along the chromatic filtration, where different stages see periodic phenomena of different periods. The simplest stage, at height 0, is rational homotopy theory. Height 1 is controlled by $K$-theory. Height 2 is controlled by elliptic cohomology and topological modular forms. Morava $E$-theory $E_n$ and Morava $K$-theory $K(n)$ play a central role at chromatic height $n$. In particular, one wishes to understand the algebraic structure found in the homotopy of commutative $E$-algebras, e.g., the $E$-cohomology of a space or the $E$-homology of an infinite loop space. Such *power operations* are analogous to the Dyer–Lashof operations acting on the homotopy of commutative $H\mathbb{F}_p$-algebras, e.g., the mod $p$ homology of an infinite loop space.

4.1. **Completed power operations for Morava $E$-theory.** Given a (commutative) $E$-algebra $X$, its homotopy $\pi_* X$ is a commutative $E_*$-algebra with additional structure. Describing all the structure is difficult, even at height $n = 1$ as in the work of J. McClure on Dyer–Lashof operations in $K$-theory. It is customary to focus on the $K(n)$-local subcategory, which is better understood. For $K(n)$-local $E$-algebras, C. Rezk constructed in [Rez09] a monad

$$\mathbb{T} \colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$$

which encodes the algebraic structure present in $\pi_* X$, i.e., such that $\pi_* X$ is naturally a $\mathbb{T}$-algebra. However, this is not the best algebraic description, as it misses the property that $\pi_* X$ is $L$-complete in the sense of Greenlees–May and Hovey–Strickland. In [BF15], we proved the following.

**Theorem 4.1.** *At every height $n$ and every prime $p$, the monad $\mathbb{T} \colon \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ canonically induces a monad $\widehat{\mathbb{T}} \colon \widehat{\mathrm{Mod}}_{E_*} \to \widehat{\mathrm{Mod}}_{E_*}$ on the subcategory of $L$-complete modules.*

If $X$ is a $K(n)$-local $E$-algebra, then $\pi_* X$ is naturally a $\widehat{\mathbb{T}}$-algebra. This way, $L$-completeness is built into the construction.

At height $n = 1$, Morava $E$-theory is $p$-completed $K$-theory: $E_1 \cong K^\wedge_p$ where $K$ denotes the complex periodic $K$-theory spectrum. For $E_1$-algebras, $K(1)$-localization is $p$-completion. Therefore, Theorem 4.1 at height $n = 1$ is about the homotopy groups of $p$-complete $K$-algebras.

4.2. **Calculations of power operations.** The monad $\mathbb{T}$ comes with a decomposition $\mathbb{T} = \bigoplus_{m \geq 0} \mathbb{T}_m$. The functor $\mathbb{T}_m$ is related to the $m^{\text{th}}$ **extended power** functor $\mathbb{P}_m(M) := (M^{\wedge_E m})_{h\Sigma_m}$, the homotopy orbit with respect to the $\Sigma_m$-action, where $M$ is any $E$-module. There is a comparison map $\alpha \colon \widehat{\mathbb{T}}_m(\pi_* M) \to \pi_* L_{K(n)} \mathbb{P}_m M$ which is an isomorphism when $M$ is a flat $E$-module. We want to construct a spectral sequence of the form

$$E^2_{s,t} = \left( (\mathbf{L}_s \widehat{\mathbb{T}}_m)(\pi_* M) \right)_t \Rightarrow \pi_{s+t} L_{K(n)} \mathbb{P}_m M$$

where $\mathbf{L}_s \widehat{\mathbb{T}}_m \colon \widehat{\mathrm{Mod}}_{E_*} \to \widehat{\mathrm{Mod}}_{E_*}$ denotes the $s^{\text{th}}$ left derived functor of the non-additive functor $\widehat{\mathbb{T}}_m$, and the comparison map $\alpha$ is an edge morphism. Following an observation by C. Rezk, the fact that $\mathbb{T}_m$ is a polynomial functor of degree $m$ will provide vanishing lines in this spectral sequence.

4.3. **Topological André–Quillen homology.** One application of Theorem 4.1 is to compute André–Quillen (co)homology in the category of $\widehat{\mathbb{T}}$-algebras. This in turn provides a way to compute topological André–Quillen (co)homology of a $K(n)$-local $E$-algebra $X$ via the algebraic André–Quillen (co)homology of its homotopy groups. More precisely, we propose to construct a spectral sequence of the form

$$\mathrm{AQ}_*(\pi_* X) \Rightarrow \mathrm{TAQ}_*(X)$$

similar to the spectral sequences studied in [Ric02].

For my prospective graduate students, this area provides interesting topics. I will help them acquire a solid background in homotopy theory, both unstable and stable, and then suggest directions to explore, notably Goerss–Hopkins obstruction theory, chromatic homotopy theory, and topological André–Quillen homology of structured ring spectra.

## 5. Motivic homotopy theory

Motivic homotopy theory applies methods from homotopy theory to algebraic geometry, more precisely to cohomology theories for schemes. Working over a base scheme $S$, the motivic stable homotopy category $\mathrm{SH}(S)$ is a triangulated category whose objects represent cohomology theories for smooth schemes over $S$. An important example is the motivic Eilenberg–MacLane spectrum $H\mathbb{Z}$, which represents motivic cohomology.

### 5.1. **The motivic Steenrod algebra.**

In his celebrated proof of the Milnor conjecture, V. Voevodsky used operations in motivic cohomology, working over a base field $\mathbb{k}$ of characteristic zero [Voe03]. Notably, he produced certain classes in the dual motivic Steenrod algebra

$$\pi_{*,*}(H\mathbb{F}_\ell \wedge H\mathbb{F}_\ell)$$

analogous to the Milnor basis, and showed that they form a basis over $\pi_{*,*}H\mathbb{F}_\ell = H^{-*,-*}(\mathbb{k};\mathbb{F}_\ell)$, the mod $\ell$ motivic cohomology of $\mathbb{k}$. In [HKØ17], it was shown that the same basis also works when $\mathbb{k}$ has positive characteristic $p \neq \ell$. In joint work with M. Spitzweck [FS18b], we prove that when $\mathbb{k}$ has characteristic $p = \ell$, the dual Steenrod algebra contains the conjectured answer as a retract. Moreover, we have reduced the conjecture to showing that over the $p$-adic integers $\mathbb{Z}_p$, the spectrum $H\mathbb{F}_p$ can be built as a homotopy colimit of dualizable objects. Another conjecture of Voevodsky states that the slices of the algebraic cobordism spectrum MGL are certain Eilenberg–MacLane spectra. We prove that the slices contain the conjectured answer as retract.

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