Equivalent statements of the telescope conjecture

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The purpose of this expository note is to clarify the relationship between various statements of the telescope conjecture. It can be viewed as a beginner's guide to the exposition in [4, §1.3]. Most of the ideas come from conversations with Charles Rezk, whom we thank for his help.

1 The statements

Throughout, p is some fixed prime and everything is localized at p.

Notation 1.1. For any (generalized) homology theory E, let L_E denote the Bousfield localization functor with respect to E, and L_E^f the finite localization functor with respect to E [5, §1].

Notation 1.2. Let L_n denote $L_{E(n)} = L_{K(0)\vee K(1)\vee \ldots \vee K(n)}$ where $E(n)$ is the Johnson-Wilson spectrum and $K(n)$ is Morava K-theory of height n.

Definition 1.3. A **type** n **complex** is a finite spectrum satisfying $K(i)_* X = 0$ for $i < n$ and $K(n)_* X \neq 0$.

By a theorem of Mitchell [6, Thm B], there exists a type n complex for every n.

Definition 1.4. Let X be a finite spectrum. A map $v\colon \Sigma^d X \to X$ is a v_n **self map** if it satisfies $K(i)_* v = 0$ for $i \neq n$ and $K(n)_* v$ is an isomorphism.

By a theorem of Hopkins-Smith [2, Thm 9], every type n complex admits a v_n self map. Note that if X has type $m > n$, then the null map $X \xrightarrow{0} X$ is a v_n self map.

Notation 1.5. Let X_n be a type n complex and $v\colon \Sigma^d X_n \to X_n$ a v_n self map. Let $\mathrm{tel}(v) = v^{-1} X_n$ denote the mapping telescope of v. By the periodicity theorem [2, Cor 3.7], $\mathrm{tel}(v)$ does not depend on the v_n self map v and we sometimes denote it $\mathrm{tel}(X_n)$ or (by abuse of notation) $\mathrm{tel}(n)$.

Here are statements of the telescope conjecture (for a given n).

(TL) Classic telescope conjecture. The map $X_n \to \mathrm{tel}(v)$ is an $E(n)$-localization (or equivalently, $K(n)$-localization) [9, 2.2].

(WK) Finite localization, weak form. The comparison map $L_n^f X_n \to L_n X_n$ is an equivalence.

(ST) Finite localization, strong form. The natural transformation $L_n^f \to L_n$ is an equivalence, i.e. the comparison map $L_n^f X \to L_n X$ is an equivalence for any X [9, 1.19 (iii)] [5, §3]. Since both L_n^f and L_n are smashing, this is the same as $L_n^f S^0 \to L_n S^0$ being an equivalence.

(BF) Bousfield classes. $\langle \mathrm{tel}(v) \rangle = \langle K(n) \rangle$ [7, 10.5].

2 The relationships

Clearly we have (ST) $\Rightarrow$ (WK).

Proposition 2.1. *(TL) $\Leftrightarrow$ (WK).*

Proof. This is [9, Thm 2.7 (iv)] or [5, Prop 14], which states that the map $X_n \to \mathrm{tel}(v)$ is a **finite** $E(n)$-localization, i.e. $\mathrm{tel}(v) = L_n^f X_n$. □

Notation 2.2. Let $\underline{F}(X, Y)$ denote the function spectrum between spectra X and Y.

Notation 2.3. Let $DX = \underline{F}(X, S^0)$ denote the Spanier-Whitehead dual of a spectrum X.

Proposition 2.4. *(BF) $\Rightarrow$ (TL).*

Proof. It suffices to show $\iota \colon X_n \to v^{-1} X_n$ is a $\mathrm{tel}(n)$-localization.

The map ι is a $\mathrm{tel}(n)$-equivalence. After smashing ι with $\mathrm{tel}(n)$, we obtain:

$$v^{-1} X_n \wedge X_n \xrightarrow{1 \wedge \iota} v^{-1} X_n \wedge v^{-1} X_n$$
$$(v \wedge 1)^{-1} (X_n \wedge X_n) \to (v \wedge v)^{-1} (X_n \wedge X_n)$$

where $X_n \wedge X_n$ is still a type n complex, $v \wedge 1$ and $v \wedge v$ are two v_n self maps, and the map (which is induced by the identity of $X_n \wedge X_n$) is therefore an equivalence, by periodicity.

The target $\mathrm{tel}(n)$ is $\mathrm{tel}(n)$-local. Let W be $\mathrm{tel}(n)$-acyclic and $f \colon W \to v^{-1} X_n$ any map. We want to show $f = 0$. Consider the square obtained by smashing f with X_n or with $v^{-1} X_n$:

$$\begin{array}{ccc} X_n \wedge W & \xrightarrow{1 \wedge f} & X_n \wedge v^{-1} X_n \\ \downarrow{\scriptstyle \iota \wedge 1} & & \simeq \downarrow{\scriptstyle \iota \wedge 1} \\ v^{-1} X_n \wedge W & \xrightarrow[1 \wedge f]{} & v^{-1} X_n \wedge v^{-1} X_n \end{array}$$

where the right-hand map is an equivalence, as shown above. The bottom left corner $v^{-1} X_n \wedge W \simeq *$ is contractible. Therefore we have $X_n \wedge f = 0$. Its adjunct map

$\eta \wedge f\colon W \to DX_n \wedge X_n \wedge v^{-1}X_n$ is also zero. But $v^{-1}X_n$ is a $(DX_n \wedge X_n)$-module spectrum (using $v^{-1}X_n = L_n^f X_n$) and f is the composite:

$$W \xrightarrow{\eta \wedge f} DX_n \wedge X_n \wedge v^{-1}X_n \to v^{-1}X_n$$

which is zero. □

Remark 2.5. Lemma [9, Lem 2.4 (i)] provides a map of abelian groups $[X_n, X_n] \to [\mathrm{tel}(X_n), \mathrm{tel}(X_n)]$ but is not quite enough to conclude that $\mathrm{tel}(X_n)$ is a module spectrum over $\underline{F}(X_n, X_n) = DX_n \wedge X_n$. Here we used the fact that L_n^f is a spectrally enriched functor, from which we obtain the map of ring spectra $\underline{F}(X_n, X_n) \to \underline{F}(L_n^f X_n, L_n^f X_n)$.

Notation 2.6. Let C_n^f denote the fiber $C_n^f \to S^0 \to L_n^f S^0$ [3, §7.3]. Warning! Our C_n^f corresponds to ΣC_n^f in [9, 2.3].

Fact 2.7. *C_{n-1}^f is a homotopy direct limit of finite complexes of type n* [9, 2.4 (iii)]. *We will write* $C_{n-1}^f = \mathrm{hocolim}_\alpha F_\alpha$.

The following lemma is also proved in [3, Prop 7.10 (d)].

Lemma 2.8. *The map $C_{n-1}^f \to S^0$ induces a natural transformation* $\mathrm{id} = \underline{F}(S^0, -) \to \underline{F}(C_{n-1}^f, -)$ *which is an X_n-localization. In particular, L_{X_n} is cosmashing.*

Proof. We want to show that for any Z, the map $Z \to \underline{F}(C_{n-1}^f, Z)$ is an X_n-localization.

The target is X_n-local. Let W be X_n-acyclic and $f\colon W \to \underline{F}(C_{n-1}^f, Z)$ any map. The map is adjunct to a map $W \wedge C_{n-1}^f \to Z$, which must be zero since the source is contractible:

$$\begin{aligned} W \wedge C_{n-1}^f &= W \wedge \left(\operatorname*{hocolim}_\alpha F_\alpha\right) \quad \text{with each } F_\alpha \text{ type } n \\ &= \operatorname*{hocolim}_\alpha (W \wedge F_\alpha) \\ &= \operatorname*{hocolim}_\alpha (*) \quad \text{because } W \text{ is } X_n\text{-acyclic} \\ &= *. \end{aligned}$$

The map is an X_n-equivalence. We have the cofiber sequence $C_{n-1}^f \to S^0 \to L_{n-1}^f S^0$ which induces a fiber sequence:

$$\underline{F}(L_{n-1}^f S^0, Z) \to \underline{F}(S^0, Z) \to \underline{F}(C_{n-1}^f, Z).$$

We want to show that the second map is an X_n-equivalence, i.e. its fiber is X_n-acyclic. We have:

$$\begin{aligned} \underline{F}(L_{n-1}^f S^0, Z) \wedge X_n &= \underline{F}(L_{n-1}^f S^0, Z \wedge X_n) \\ &= \underline{F}(L_{n-1}^f S^0, Z \wedge DDX_n) \quad \text{since } X_n \text{ is finite} \\ &= \underline{F}(L_{n-1}^f S^0 \wedge DX_n, Z) \text{ since } DX_n \text{ is equivalent to finite} \\ &= \underline{F}(L_{n-1}^f (DX_n), Z) \quad \text{since } L_{n-1}^f \text{ is smashing.} \end{aligned}$$

Suffices to check $L^f_{n-1}(DX_n) \simeq *$. We know DX_n is finite of type n, that is $K(0) \vee \ldots \vee K(n-1)$-acyclic. But since it is finite, it is also finitely $K(0) \vee \ldots \vee K(n-1)$-acyclic and so we have $L^f_{n-1}(DX_n) \simeq *$. □

Lemma 2.9. *Let X be a type n complex. Then its Spanier-Whitehead dual DX is also (equivalent to) a type n complex.*

Proof. Since X is finite, DX is equivalent to a finite complex [1, Lem III.5.5]. Since X is finite, we have:

$$\begin{aligned} K(i)_* DX &\cong K(i)^{-*} X \\ &\cong \mathrm{Hom}_{K(i)_*}(K(i)_* X, K(i)_*) \end{aligned}$$

which is zero whenever $K(i)_* X$ is zero, in particular for $i < n$.

On the other hand, [2, Lem 1.13] says that $K(i)_* X$ is non-zero if and only if the duality map $S^0 \to X \wedge DX$ is non-zero in $K(i)$-homology. In particular, we then have $0 \neq K(i)_*(X \wedge DX) \cong K(i)_* X \otimes_{K(i)_*} K(i)_* DX$ which guarantees $K(i)_* DX \neq 0$. In the case at hand, we have $K(n)_* X \neq 0$ which means $K(n)_* DX \neq 0$ and so DX has type n. □

Proposition 2.10. *(WK) ⇒ (BF).*

Here are two different proofs, relying on different facts.

Fact 2.11. *There are factorizations of localization functors:*

1. $L_{K(n)} = L_{X_n} L_n$
2. $L_{\mathrm{tel}(n)} = L_{X_n} L^f_n$.

First proof of 2.10. We want to show $L_{\mathrm{tel}(n)} Y \to L_{K(n)} Y$ is an equivalence for any Y. The source and target can be computed using 2.11. Using 2.8 and 2.7, for any Z we have:

$$\begin{aligned} L_{X_n} Z &= \underline{F}(C^f_{n-1}, Z) \\ &= \underline{F}(\operatorname*{hocolim}_\alpha F_\alpha, Z) \\ &= \operatorname*{holim}_\alpha \underline{F}(F_\alpha, Z) \\ &= \operatorname*{holim}_\alpha \underline{F}(F_\alpha, S^0) \wedge Z \\ &= \operatorname*{holim}_\alpha (DF_\alpha \wedge Z). \end{aligned}$$

Using this, we can compare $L_{\mathrm{tel}(n)}Y$ and $L_{K(n)}Y$:

$$\begin{aligned}
L_{\mathrm{tel}(n)}Y &= L_{X_n}L_n^f Y \\
&= \operatorname{holim}_{\alpha}(DF_\alpha \wedge L_n^f Y) \\
&= \operatorname{holim}_{\alpha}\left(L_n^f(DF_\alpha) \wedge Y\right) \text{ since } L_n^f \text{ is smashing} \\
&= \operatorname{holim}_{\alpha}\left(L_n(DF_\alpha) \wedge Y\right) \text{ by (WK) and lemma 2.9} \\
&= \operatorname{holim}_{\alpha}(DF_\alpha \wedge L_n Y) \text{ since } L_n \text{ is smashing} \\
&= L_{X_n}L_n Y \\
&= L_{K(n)}Y.
\end{aligned}$$

□

Fact 2.12. *For all finite complexes X of type at least n (i.e. $K(n-1)$-acyclic), we have $\langle L_n X\rangle \leq \langle K(n)\rangle$.*

The collection of all such X is thick, so it suffices to build one example, which is done in [8, §8.3]. Using this fact, we obtain an alternate proof of 2.10, outlined in [4, 1.13 (ii)].

Second proof of 2.10. Our assumption says $\mathrm{tel}(X_n) = L_n X_n$ and we want to show the equality $\langle \mathrm{tel}(X_n)\rangle = \langle K(n)\rangle$, that is $\langle L_n X_n\rangle = \langle K(n)\rangle$. By 2.12, it suffices to show $\langle L_n X_n\rangle \geq \langle K(n)\rangle$.

Let W be $L_n X_n$-acyclic. We want to show W is $K(n)$-acyclic. We know $* \simeq W \wedge L_n X_n = L_n(W \wedge X_n)$ so that $W \wedge X_n$ is $K(0) \vee \ldots \vee K(n)$-acyclic, and in particular $K(n)$-acyclic. That means we have:

$$K(n)_*(W \wedge X_n) = 0 = K(n)_* W \otimes_{K(n)_*} K(n)_* X_n$$

which forces $K(n)_* W$ to be zero since X_n has type n. □

Remark 2.13. If one is willing to use the fact 2.11, then the implication (ST) ⇒ (BF) is immediate, without going through (ST) ⇒ (WK) ⇒ (BF). The assumption $L_n^f = L_n$ yields:

$$L_{\mathrm{tel}(n)} = L_{X_n}L_n^f = L_{X_n}L_n = L_{K(n)}$$

so that $K(n)$ and $\mathrm{tel}(n)$ are Bousfield equivalent.

Proposition 2.14. *(WK) at n and (ST) at $n-1$ ⇒ (ST) at n.*

In particular: [(WK) from 0 to n] ⇒ [(ST) from 0 to n] since (ST) is true at $n = 0$.

Here is a proof proposed in [4, 1.13 (iv)].

First proof of 2.14. Consider the cofiber sequence $C_{n-1}^f \to S^0 \to L_{n-1}^f S^0$ where the fiber C_{n-1}^f is a homotopy direct limit of finite type n complexes $\operatorname{hocolim}_\alpha F_\alpha$. Applying

L_n^f or L_n, we obtain a map of cofiber sequences:

$$\begin{array}{ccccc}
L_n^f C_{n-1}^f & \longrightarrow & L_n^f S^0 & \longrightarrow & L_n^f L_{n-1}^f S^0 = L_{n-1}^f S_0 \\
\downarrow & & \downarrow & & \downarrow \\
L_n C_{n-1}^f & \longrightarrow & L_n S^0 & \longrightarrow & L_n L_{n-1}^f S^0.
\end{array}$$

Since L_n and L_n^f are smashing, they commute with homotopy direct limits and we have:

$$\begin{aligned}
L_n^f C_{n-1}^f &= L_n^f \operatorname{hocolim}_\alpha F_\alpha \\
&= \operatorname{hocolim}_\alpha L_n^f F_\alpha \\
&= \operatorname{hocolim}_\alpha L_n F_\alpha \quad \text{by (WK)} \\
&= L_n \operatorname{hocolim}_\alpha F_\alpha \\
&= L_n C_{n-1}^f
\end{aligned}$$

so that the left-hand downward map is an equivalence.

By inductive assumption (ST) at $n-1$, we have $L_{n-1}^f = L_{n-1}$ so that the bottom right corner is $L_n L_{n-1}^f S^0 = L_n L_{n-1} S^0 = L_{n-1} S^0$ and the right-hand downward map is the equivalence $L_{n-1}^f S_0 \xrightarrow{\simeq} L_{n-1} S^0$. Therefore the middle downward map is an equivalence, which is the statement of (ST) at n. □

Here is an alternate proof of 2.14, which can be restated as: (BF) at n and (ST) at $n-1 \Rightarrow$ (ST) at n. It will rely on the relationship between L_n and L_{n-1}.

Fact 2.15. *(Fracture squares) The squares:*

$$\begin{array}{ccc}
L_n & \longrightarrow & L_{K(n)} \\
\downarrow & \lrcorner & \downarrow \\
L_{n-1} & \longrightarrow & L_{n-1} L_{K(n)}
\end{array}$$

$$\begin{array}{ccc}
L_n^f & \longrightarrow & L_{\mathrm{tel}(n)} \\
\downarrow & \lrcorner & \downarrow \\
L_{n-1}^f & \longrightarrow & L_{n-1}^f L_{\mathrm{tel}(n)}.
\end{array}$$

are homotopy pullbacks.

Second proof of 2.14. The assumptions say that the bottom rows and right-hand sides of the two fracture squares 2.15 are equivalent, hence so are the top left corners $L_n^f \simeq L_n$. □

Remark 2.16. In the proof, we only needed (BF) on S^0, i.e. that $L_{\mathrm{tel}(n)} S^0 \to L_{K(n)} S^0$ be an equivalence.

Lemma 2.17. *The full subcategory of the (homotopy) category of (p-local) spectra X for which the telescope conjecture holds, i.e.* $L_n^f X \xrightarrow{\simeq} L_n X$ *is thick.*

Proof. This follows from the natural transformation $L_n^f \to L_n$ and the fact that both functors preserve cofiber sequences.

More precisely, assume Y is a retract of X and $L_n^f X \to L_n X$ is an equivalence. Then naturality makes $L_n^f Y \to L_n Y$ a retract of $L_n^f X \to L_n X$ and hence an equivalence.

Assume $X \to Y \to Z$ is a cofiber sequence such that $L_n^f \to L_n$ is an equivalence for two of the three. Then the map of cofiber sequences:

$$\begin{array}{ccccc} L_n^f X & \longrightarrow & L_n^f Y & \longrightarrow & L_n^f Z \\ \downarrow & & \downarrow & & \downarrow \\ L_n X & \longrightarrow & L_n Y & \longrightarrow & L_n Z \end{array}$$

makes the third comparison map an equivalence. □

Fact 2.18. *The telescope conjecture (ST) holds (at all n) on all* L_i*-local spectra, for any i. That is, for any spectrum X and integers* $n, i \geq 0$*, we have* $L_n^f L_i X \xrightarrow{\simeq} L_n L_i X$.

Proposition 2.19. *(ST) at* $n \Rightarrow$ *(ST) at* $n-1$.

Proof. By assumption the map $L_n^f S^0 \xrightarrow{\simeq} L_n S^0$ is an equivalence. Applying L_{n-1}^f, we obtain the equivalence:

$$L_{n-1}^f L_n^f S^0 \xrightarrow{\simeq} L_{n-1}^f L_n S^0.$$

The left-hand side is $L_{n-1}^f S^0$. By 2.18, the right-hand side is $L_{n-1}^f L_n S^0 \simeq L_{n-1} L_n S^0 = L_{n-1} S^0$. □

In particular, (ST) at $n \Leftrightarrow$ (ST) from 0 to $n \Rightarrow$ (WK) from 0 to n and by 2.14, the converse holds as well. In summary, the "strong" telescope conjecture (ST) is simply the conjunction of the "weak" telescope conjectures (WK=TL=BF) from 0 to n.

Here is another equivalent statement of the telescope conjecture, arguably the most important.

(AN) Adams-Novikov spectral sequence. The Adams-Novikov spectral sequence for $\mathrm{tel}(X_n)$ converges to $\pi_* \mathrm{tel}(X_n)$.

We refer to [4, 1.13 (iii)] for more information.

3 Proving some of the facts

Proof of the fracture squares 2.15. See [3, Thm 6.19]. Indeed, we want to show that for any Z, the square:

$$\begin{array}{ccc} L_n Z & \longrightarrow & L_{K(n)} Z \\ \downarrow & \lrcorner & \downarrow \\ L_{n-1} Z & \longrightarrow & L_{n-1} L_{K(n)} Z \end{array}$$

is a homotopy pullback. Note the equivalence $L_{K(n)} = L_{K(n)}L_n = L_n L_{K(n)}$, since $K(n)$-local spectra are in particular L_n-local. The claim is equivalent to the induced map of vertical fibers (monochromatic layers) $M_n Z \to M_n L_{K(n)} Z$ being an equivalence, which is proved in [3, Thm 6.19].

Here is an alternate proof proposed by C. Rezk. First note that all four corners are L_n-local, because they are local with respect to some subwedge of $K(0) \vee \ldots \vee K(n)$. Therefore the vertical (or horizontal) fibers are also L_n-local and the induced map will be an equivalence if and only if it is a $K(0) \vee \ldots \vee K(n)$-equivalence. In other words, it suffices to check that the square is a homotopy pullback after smashing with $K(i)$, for $i = 0, \ldots, n$.

For $i = 0, \ldots, n-1$, the vertical maps are $K(i)$-equivalences, since they are the unit $\mathrm{id} \to L_{n-1}$ applied to something, respectively L_n and $L_{K(n)}$. Thus after smashing with $K(i)$, the square is a homotopy pullback with vertical fibers $\simeq *$.

For $i = n$, the top map is a $K(n)$-equivalence since it is the unit $\mathrm{id} \to L_{K(n)}$ applied to L_n. The bottom objects are $K(n)$-acyclic, as we show below. Thus after smashing with $K(n)$, the square is a homotopy pullback with horizontal fibers $\simeq *$.

Why is $L_{n-1}Z$ $K(n)$-acyclic? Pick a type n complex X_n. Since X_n and DX_n are $K(0) \wedge \ldots K(n-1)$-acyclic (by 2.9), we have:

$$X_n \wedge L_{n-1}Z = \underline{F}(DX_n, L_{n-1}Z) = *.$$

In $K(n)$-homology, we obtain:

$$0 = K(n)_*(X_n \wedge L_{n-1}Z) = K(n)_* X_n \otimes_{K(n)_*} K(n)_* L_{n-1}Z$$

which forces $K(n)_* L_{n-1}Z = 0$ since we have $K(n)_* X_n \neq 0$. This concludes the proof of the first fracture square, and the second has a similar proof. □

Proof of the factorizations 2.11. 1. [3, Prop 7.10 (e)].
2. Similar. □

Proof of the telescope conjecture on L_i-local spectra 2.18. See [3, Cor 6.10]. Here is a sketch of an alternate proof.

First, the telescope conjecture holds (at all n) for MU, meaning $L_n^f MU \xrightarrow{\simeq} L_n MU$ is an equivalence [9, Thm 2.7 (iii)].

Second, the telescope conjecture holds for spectra of the form $MU \wedge Z$ for any Z. This is clear once we know L_n is smashing:

$$\begin{aligned} L_n^f(MU \wedge Z) &= L_n^f MU \wedge Z \\ &= L_n MU \wedge Z \\ &= L_n(MU \wedge Z). \end{aligned}$$

In fact, one can show directly the property: $L_n(MU \wedge Z) \simeq L_n MU \wedge Z$ without knowing that L_n is smashing. In particular, the telescope conjecture holds for all spectra of the form $L_n MU \wedge Z$.

By lemma 2.17, the telescope conjecture holds on the thick subcategory of the (homotopy) category of (p-local) spectra generated by spectra of the form $L_n MU \wedge Z$

for arbitrary Z. Call this thick subcategory $\mathcal{T}$. Note that $\mathcal{T}$ is closed under smashing by anything.

To prove the claim, it suffices to show that for any X, L_nX is in $\mathcal{T}$. It suffices to show L_nS^0 is in $\mathcal{T}$, because of the equivalence $L_nX = X \wedge L_nS^0$. Using the (MU-based) Adams-Novikov resolution of S^0 and arguments similar to [3, Pf of Prop 6.5], one can show L_nS^0 is in $\mathcal{T}$. ◻

In the statement of (TL), we used the following fact, which is a particular case of [3, Lem 7.2].

Proposition 3.1. *$K(n)$- and $E(n)$-localizations agree on any type n complex: $L_nX_n = L_{K(n)}X_n$.*

Proof. We show that the map $X_n \to L_nX_n$ is in fact a $K(n)$-localization. We know it is a $K(0) \vee \ldots \vee K(n)$-equivalence, in particular a $K(n)$-equivalence.

Remains to show that the target L_nX_n is $K(n)$-local. We have $L_nX_n = X_n \wedge L_nS^0$ (just because X_n is finite and localizations preserve cofiber sequences [7, Prop 1.6]; no need to invoke the fact that L_n is smashing). Let W be $K(n)$-acyclic. We want to show $[W, X_n \wedge L_nS^0] = 0$. But we have $[W, X_n \wedge L_nS^0] = [W \wedge DX_n, L_nS^0] = 0$ since $W \wedge DX_n$ is $K(0) \vee \ldots \vee K(n)$-acyclic. Indeed, we have $K(n)_*W = 0$ by assumption and $K(i)_*(DX_n) = 0$ for $i = 0, \ldots, n-1$ since DX_n has type n, and thus $K(i)_*(W \wedge DX_n) = K(i)_*W \otimes_{K(i)_*} K(i)_*(DX_n) = 0$ for $i = 0, \ldots, n$. ◻

Notation 3.2. Let Thick(X) denote the thick subcategory generated by a spectrum X. Similar notation for a set of spectra.

Lemma 3.3. *If Z is in* Thick(Y)*, then we have $\langle Z \rangle \leq \langle Y \rangle$.*

Proof. Clearly we have $\langle Y \rangle \leq \langle Y \rangle$.

If Z_1 is a retract of Z_2 and W is Z_2-acyclic, then $Z_1 \wedge W$ is a retract of $Z_2 \wedge W \simeq *$ and so is contractible. In other words, W is also Z_1-acyclic, and we have $\langle Z_1 \rangle \leq \langle Z_2 \rangle$.

If $Z_1 \to Z_2 \to Z_3$ is a cofiber sequence and i, j, k is a permutation of $1, 2, 3$ satisfying $\left\langle Z_j \right\rangle \leq \langle Y \rangle$ and $\langle Z_k \rangle \leq \langle Y \rangle$, then we have $\langle Z_i \rangle \leq \left\langle Z_j \right\rangle \vee \langle Z_k \rangle \leq \langle Y \rangle \vee \langle Y \rangle = \langle Y \rangle$ [7, Prop 1.23]. ◻

In the statement of (BF), we used the following fact.

Proposition 3.4. *n-telescopes are all Bousfield equivalent. In other words, if X_n and Y_n are type n complexes with v_n self maps v and w, then we have $\left\langle v^{-1}X_n \right\rangle = \left\langle w^{-1}Y_n \right\rangle$.*

Proof. X_n and Y_n generate the same thick subcategory and so are Bousfield equivalent by 3.3. Let us show that the telescopes $v^{-1}X_n$ and $w^{-1}Y_n$ also generate the same thick subcategory.

If Z_1 is a retract of Z_2 and the latter is in Thick(Y_n) (i.e. finite complexes of type at least n) and such that its telescope tel(Z_2) is in Thick(tel(Y_n)), then tel(Z_1) = $L_n^f Z_1$ is a retract of tel(Z_2) = $L_n^f Z_2$ and so is in Thick(tel(Y_n)).

If $Z_1 \to Z_2 \to Z_3$ is a cofiber sequence where two of the objects are in Thick(Y_n) and such that their telescopes are in Thick(tel(Y_n)), then telescoping yields the cofiber sequence $L_n^f Z_1 \to L_n^f Z_2 \to L_n^f Z_3$ so that the telescope $\mathrm{tel}(Z_i)$ of the third object Z_i is also in $\mathrm{Thick}(\mathrm{tel}(Y_n))$.

Since X_n is in $\mathrm{Thick}(Y_n)$, the above discussion shows $\mathrm{tel}(X_n)$ is in $\mathrm{Thick}(\mathrm{tel}(Y_n))$. □

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