Eilenberg–MacLane mapping algebras and higher distributivity up to homotopy

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Homotopy Theory: Tools and Applications
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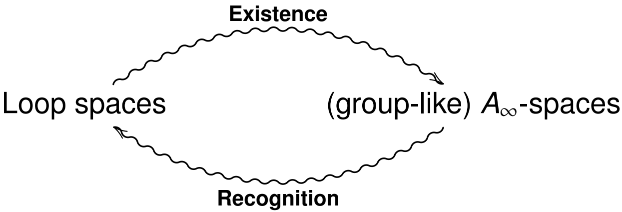
Outline

Some history: A_∞-spaces

Stasheff (1963): Higher associativity via associahedra.



- **Homotopy invariance**: Assume $X \simeq Y$. Then X admits an A_n-structure if and only if Y does.
- **Strictification**: An A_∞-space is weakly equivalent to a topological monoid.

Stable cohomology operations

Slogan: Higher distributivity via cubes.

Let X be a spectrum. Cohomology $H^n(X;\mathbb{F}_p) = [X, \Sigma^n H\mathbb{F}_p]$ is given by homotopy classes of maps to Eilenberg–MacLane spectra.

Primary stable cohomology operations are given by homotopy classes of maps between Eilenberg–MacLane spectra.

The mod p Steenrod algebra $\mathcal{A}^*$ is given by

$$\mathcal{A}^k = [H\mathbb{F}_p, \Sigma^k H\mathbb{F}_p].$$

For example, $\mathrm{Sq}^k \colon H\mathbb{F}_2 \to \Sigma^k H\mathbb{F}_2$.

More generally, consider maps between finite products

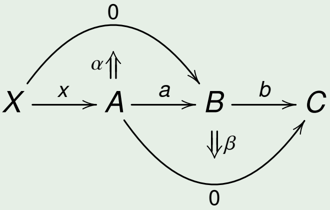
$$A = \Sigma^{n_1} H\mathbb{F}_p \times \ldots \times \Sigma^{n_k} H\mathbb{F}_p.$$

Higher order operations

Higher order cohomology operations are encoded by the *mapping spaces* between Eilenberg–MacLane spectra.

Example

The 3-fold Toda brackets $\langle b, a, x \rangle \subseteq [X, \Omega C]$ define a secondary cohomology operation $\langle b, a, - \rangle$.



Distributivity up to homotopy

In the homotopy category of spectra, composition is bilinear.

This does *not* hold in a $\mathbf{Top}_*$-enriched category of spectra.

$$X \xrightarrow{x,x'} A \xrightarrow{a,a'} B$$

The equation

$$(a + a')x = ax + a'x$$

holds strictly in $\mathrm{map}(X, B)$, because of pointwise addition. That is, **left linearity** holds.

The equation

$$a(x + x') \sim ax + ax'$$

holds up to *coherent homotopy* in $\mathrm{map}(X, B)$.

Goal

Describe the higher distributivity laws satisfied by maps between Eilenberg–MacLane spectra.

Maps between Eilenberg–MacLane spectra

Work in a simplicial model category of spectra **Sp** (e.g. Bousfield–Friedlander).

Important ingredient: A model of the Eilenberg–MacLane spectrum $H\mathbb{F}_p$ which is an abelian group object, fibrant, and cofibrant. (Hat tip: Marc Stephan.)

$\Rightarrow$ Each mapping space map(X, A) is a topological abelian group.

Notation

Let $\mathcal{EM}$ denote the full $\mathbf{Top}_*$-enriched category of **Sp** consisting of the finite products

$$A = \Sigma^{n_1} H\mathbb{F}_p \times \ldots \times \Sigma^{n_k} H\mathbb{F}_p.$$

Note that $\mathcal{EM}$ is a small category.

Left linear mapping theories

Salient features of $\mathcal{EM}$:

1. **Top**$_*$-enriched.
2. Has finite products (i.e., is a *theory*).
3. Each mapping space $\mathcal{EM}(A, B)$ is an topological abelian group, with basepoint $0\colon A \to B$.
4. Composition is strictly left linear: $(a + a')x = ax + a'x$.

Definition

A **left linear mapping theory** $\mathcal{T}$ is defined by (1)–(4).

Examples

Example

$\mathcal{EM}$ is a left linear mapping theory, called the **Eilenberg–MacLane mapping theory.**

Example

Consider models of Eilenberg–MacLane spaces $K(\mathbb{F}_p, n)$ as topological abelian groups. Let $\mathcal{EM}^{\text{unstable}}$ be the full subcategory of $\mathbf{Top}_*$ consisting of finite products

$$K(\mathbb{F}_p, n_1) \times \ldots \times K(\mathbb{F}_p, n_k)$$

with $n_i \geq 1$. Then $\mathcal{EM}^{\text{unstable}}$ is a left linear mapping theory.

Enriched cohomology

For a spectrum X, the functor

$$\text{map}(X, -) \colon \mathcal{EM} \to \textbf{Top}_*$$

preserves products strictly, i.e., is a model of $\mathcal{EM}$. It is called the (stable) **Eilenberg–MacLane mapping algebra** of X.

For our purposes: It suffices to focus on $\mathcal{EM}$ itself.

What about right linearity?

Stably, finite products become coproducts, more precisely in the *homotopy category* of spectra:

$$A \vee B \xrightarrow{\cong} A \times B.$$

In spectra, finite products are *weak* coproducts:

$$A \vee B \xrightarrow{\sim} A \times B.$$

Definition

A mapping theory $\mathcal{T}$ is **weakly bilinear** if it is left linear and moreover for all objects A, B, Z of $\mathcal{T}$, the map

$$\mathcal{T}(A \times B, Z) \xrightarrow{(i_A^*, i_B^*)} \mathcal{T}(A, Z) \times \mathcal{T}(B, Z)$$

is a trivial Serre fibration.

Examples

Example

The mapping theory $\mathcal{EM}$ is weakly bilinear.

Example

The mapping theory $\mathcal{EM}^{\text{unstable}}$ is left linear, but *not* weakly bilinear.

1-distributivity

Definition

A left linear **Top**$_*$-enriched category is 1-**distributive** if for all $a, x, y \in \mathcal{T}$, there is a path

$$a(x+y) \xrightarrow[\varphi_a^{x,y}]{} ax + ay$$

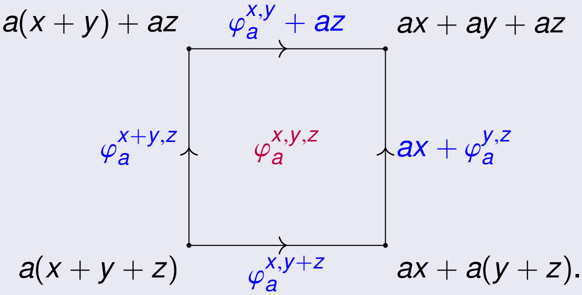
in $\mathcal{T}$. In other words, $\mathcal{T}$ is right linear up to homotopy.
A choice of such paths is denoted $\varphi^1 = \left\{\varphi_a^{x,y} \mid a, x, y \in \mathcal{T}\right\}$ and is called a 1-**distributor** for $\mathcal{T}$.
Also, φ^1 is required to be continuous in the inputs a, x, y. More precisely, for all objects X, A, B of $\mathcal{T}$, the following map is continuous:

$$\mathcal{T}(A,B) \times \mathcal{T}(X,A)^2 \xrightarrow{\varphi^1} \mathcal{T}(X,B)^I$$
$$(a, x, y) \longmapsto \varphi_a^{x,y}.$$

2-distributivity

Definition

$\mathcal{T}$ is called **2-distributive** if it admits a 1-distributor φ^1 such that for all $a, x, y, z \in \mathcal{T}$, the map $\partial I^2 \to \mathcal{T}$ defined by



admits an extension $\varphi_a^{x,y,z} \colon I^2 \to \mathcal{T}$.

2-distributivity (cont'd)

Definition

A choice of such 2-cubes is denoted

$$\varphi^2 = \left\{ \varphi_a^{x,y,z} \mid a, x, y, z \in \mathcal{T} \right\}$$

and is called a 2-**distributor** for $\mathcal{T}$, **based** on the 1-distributor φ^1.

As before, the 2-distributor φ^2 is required to be continuous in the inputs $a, x, y, z \in \mathcal{T}$. More precisely, for all objects X, A, B of $\mathcal{T}$, the following map is continuous:

$$\begin{aligned} \mathcal{T}(A,B) \times \mathcal{T}(X,A)^3 &\xrightarrow{\varphi^2} \mathcal{T}(X,B)^{I^2} \\ (a,x,y,z) &\longmapsto \varphi_a^{x,y,z}. \end{aligned}$$

n-distributivity

Definition

$\mathcal{T}$ is called ***n*-distributive** if there are collections of cubes $\varphi^0, \varphi^1, \ldots, \varphi^n$, where

$$\varphi^m = \{\varphi_a^{x_0,\ldots,x_m} \mid a, x_0, \ldots, x_m \in \mathcal{T}\}$$

is a collection of m-cubes $\varphi_a^{x_0,\ldots,x_m} \colon I^m \to \mathcal{T}$, satisfying the following.

- φ^0 is a 0-distributor, i.e., the collection of 0-cubes $\varphi_a^x = ax$.
- For $1 \le m \le n$, the following boundary conditions hold:

$$\varphi_a^{x_0,\ldots,x_m}(t_1,\ldots,\overbrace{0}^{t_j},\ldots,t_m) = \varphi_a^{x_0,\ldots,x_{j-1}+x_j,\ldots,x_m}(t_1,\ldots,\widehat{t_j},\ldots,t_m)$$

$$\varphi_a^{x_0,\ldots,x_m}(t_1,\ldots,\overbrace{1}^{t_j},\ldots,t_m) = \varphi_a^{x_0,\ldots,x_{j-1}}(t_1,\ldots,\ldots,\ldots,t_{j-1}) \oplus \varphi_a^{x_j,\ldots,x_m}(t_{j+1},\ldots,t_m).$$

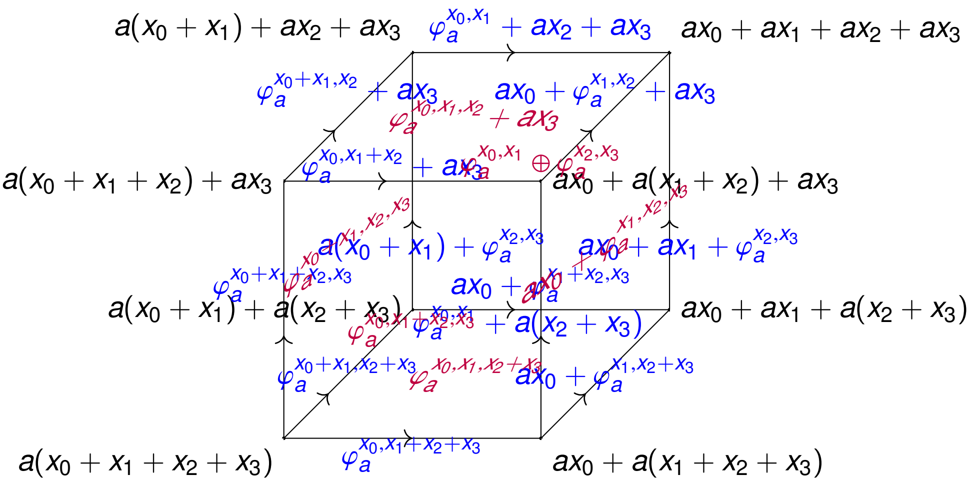
n-distributivity (cont'd)

Such a collection φ^n of n-cubes in $\mathcal{T}$ is called an n-**distributor** for $\mathcal{T}$, **based** on the $(n-1)$-distributor φ^{n-1}.

The n-distributor φ^n is required to be continuous in the inputs $a, x_0, \ldots, x_n \in \mathcal{T}$. More precisely, for all objects X, A, B of $\mathcal{T}$, the following map is continuous:

$$\begin{aligned} \mathcal{T}(A,B) \times \mathcal{T}(X,A)^{n+1} &\xrightarrow{\varphi^n} \mathcal{T}(X,B)^{I^n} \\ (a, x_0, \ldots, x_n) &\longmapsto \varphi_a^{x_0,\ldots,x_n}. \end{aligned}$$

Example: 3-distributor



Existence

Theorem (Baues,F.)

Let $\mathcal{T}$ be a weakly bilinear mapping theory in which every mapping space $\mathcal{T}(A, B)$ has the homotopy type of a CW complex. Then $\mathcal{T}$ is ∞-distributive.

Remark

In fact, $\mathcal{T}$ admits a "good" ∞-distributor, for which each distributor φ^n is determined up to homotopy rel ∂I^n by the previous distributor φ^{n-1}.

Homotopy invariance

Theorem (Baues,F.)

Let $F\colon \mathcal{S} \to \mathcal{T}$ be a morphism of left linear $\mathbf{Top}_*$*-enriched categories which is moreover a Dwyer–Kan equivalence.*
Assume that all mapping spaces in $\mathcal{S}$ and in $\mathcal{T}$ have the homotopy type of a CW complex. Then for every $n \geq 1$ (or $n = \infty$), $\mathcal{S}$ is n-distributive if and only if $\mathcal{T}$ is n-distributive.

The Kristensen derivation

Fix $p = 2$, and let φ^1 be a "good" 1-distributor for the mod 2 Eilenberg–MacLane mapping theory $\mathcal{EM}$.

For a class in the Steenrod algebra $a \in \mathcal{A}^m$, the loop

$$0 = a0 = a(1+1) \xrightarrow{\varphi_a^{1,1}} a1 + a1 = a + a = 0$$

defines a class

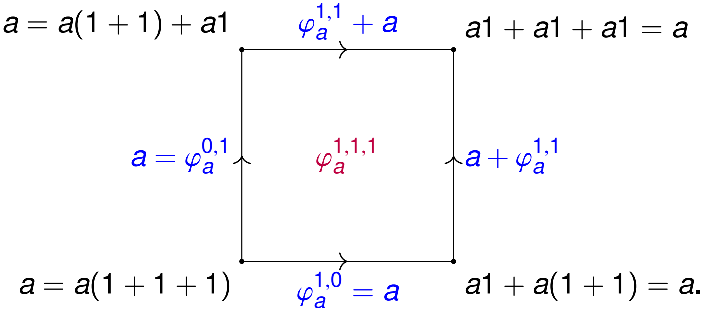
$$\kappa(a) \in \pi_1 \mathcal{EM}(H\mathbb{F}_2, \Sigma^m H\mathbb{F}_2) = [H\mathbb{F}_2, \Sigma^{m-1} H\mathbb{F}_2] = \mathcal{A}^{m-1}.$$

Proposition (Baues 2006)

The function $\kappa \colon \mathcal{A}^* \to \mathcal{A}^{*-1}$ is the Kristensen derivation, i.e., the derivation satisfying $\kappa(\mathrm{Sq}^m) = \mathrm{Sq}^{m-1}$.

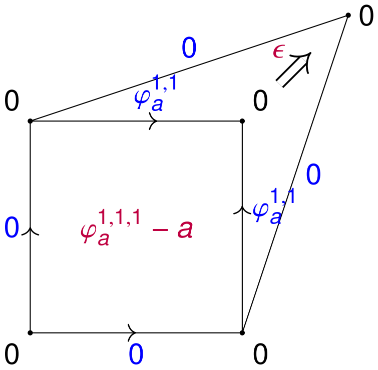
The 2-dimensional analogue

Now let φ^2 be a "good" 2-distributor for $\mathcal{EM}$. Consider the 2-cube in $\mathcal{EM}(H\mathbb{F}_2, \Sigma^m H\mathbb{F}_2)$:



A derivation of degree –2

The 2-cube



defines a class

$$\lambda(a) \in \pi_2 \mathcal{E}\mathcal{M}(H\mathbb{F}_2, \Sigma^m H\mathbb{F}_2) = \mathcal{A}^{m-2}.$$

A derivation of degree –2 (cont'd)

Proposition

The function $\lambda\colon \mathcal{A}^* \to \mathcal{A}^{*-2}$ *is a derivation.*

Question

Is λ given by $\lambda = \kappa^2$?

Thank you!

Reference

- H.J. Baues and M. Frankland. Eilenberg–MacLane mapping algebras and higher distributivity up to homotopy. arXiv:1703.07512.