Realization problems in algebraic topology

Martin Frankland

University of Western Ontario

mfrankla@uwo.ca

Wayne State University
Mathematics Colloquium
November 3, 2014

Outline

Algebraic invariants

$\boxed{\text{Topology}} \rightsquigarrow \boxed{\text{Algebra}}$

Let X be a space.

- $H^*(X; \mathbb{F}_p)$ is an unstable algebra over the Steenrod algebra $\mathcal{A}$.
- $H_*(X; \mathbb{F}_p)$ is an unstable coalgebra over $\mathcal{A}$.
- $\pi_* X$ is a Π-algebra, i.e., graded group with action of primary homotopy operations.

Let X be a spectrum and R a ring spectrum, e.g., $R = H\mathbb{F}_p$.

- $R^* X$ is an $R^* R$-module.
- $R_* X$ is an $R_* R$-comodule.
- $\pi_* X$ is a π_*^S-module, where $\pi_*^S = \pi_*(S)$ is the stable homotopy ring.

Π-algebras

Π-algebra ≈ graded group with additional structure which looks like the homotopy groups of a space.

Definition

- Π := full subcategory of the homotopy category of pointed spaces consisting of finite wedges of spheres $\bigvee S^{n_i}$, $n_i \geq 1$.
- **Π-algebra** := product-preserving functor $A \colon \Pi^{\mathrm{op}} \to \mathbf{Set}_*$.

Example

$\pi_* X = [-, X]_*$ for a pointed space X.

Notation

Write $A_n := A(S^n)$.

Realizations

Realization Problem

Given a Π-algebra A, is there a space X satisfying $\pi_* X \simeq A$ as Π-algebras?

Classification Problem

If A is realizable, can we **classify** all realizations?

Some examples

- Simplest Π-algebras: Only one non-trivial group A_n.
- Answer: Always realizable (uniquely), by an Eilenberg-Maclane space $K(A_n, n)$.
- Next simplest case: Only 2 non-trivial groups A_n, A_{n+k}. Assume $n \geq 2$.
- Answer: **Not** always realizable...

Warm-up

Case $k = 1$: Always realizable (classic).
Case $k = 2$: Always realizable (a bit of work).

Classify?

- Naive: List of realizations $= \pi_0 \mathcal{T}\mathcal{M}(A)$.
- Better: **Moduli space** $\mathcal{T}\mathcal{M}(A)$ of realizations.

Remark

Relative moduli space $\mathcal{T}\mathcal{M}'(A)$*: Realizations* X *with identification* $\pi_* X \simeq A$*. Have fiber sequence:*

$$\mathcal{T}\mathcal{M}'(A) \xrightarrow{\text{forget}} \mathcal{T}\mathcal{M}(A) \to B\operatorname{Aut}(A)$$

and $\mathcal{T}\mathcal{M}(A) \simeq \mathcal{T}\mathcal{M}'(A)_{h\operatorname{Aut}(A)}$.

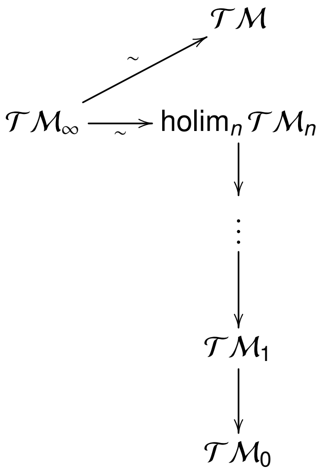
Moduli Space

$\mathcal{TM}(A) =$ nerve of the category with

- Objects: Realizations X.
- Morphisms: Weak equivalences $X \to X'$.

$$\mathcal{TM}(A) \simeq \coprod_{\langle X \rangle} B\,\mathrm{Aut}^h(X).$$

- Blanc–Dwyer–Goerss (2004): Obstruction theory for building $\mathcal{TM}(A)$.
- Successive approximations $\mathcal{TM}_n(A)$, $0 \leq n \leq \infty$.



Building $\mathcal{TM}(A)$

- $\mathcal{TM}_0(A) \simeq B\,\mathrm{Aut}(A)$.
- $\mathcal{TM}_n(A) \to \mathcal{TM}_{n-1}(A)$ related by a fiber square.
- For Y in $\mathcal{TM}_{n-1}$ and $\mathcal{M}(Y) \subseteq \mathcal{TM}_{n-1}$ its component, we have:

$$\mathcal{H}^{n+1}(A; \Omega^n A) \to \mathcal{TM}_n(A)_Y \to \mathcal{M}(Y)$$

  where fiber $=$ Quillen cohomology "space".

- Obstruction to lifting $\in \mathrm{HQ}^{n+2}(A; \Omega^n A)$
- Lifts classified by $\pi_0(\text{fiber}) = \mathrm{HQ}^{n+1}(A; \Omega^n A)$.

Problem

Can we compute the obstruction groups?

Beck modules

Definition

Let $\mathcal{C}$ be an algebraic category and X an object in $\mathcal{C}$. A (Beck) **module** over X is an abelian group object in the slice category over X:

$$(\mathcal{C}/X)_{\mathrm{ab}}.$$

Example

$\mathcal{C} =$ Groups. A Beck module over G is a split extension:

$$G \ltimes M \twoheadrightarrow G.$$

Note: $(g, m)(g', m') = (gg', m + gm')$.

Beck modules (cont'd)

Example

$\mathcal{C} =$ Commutative rings. A Beck module over R is a square-zero extension:

$$R \oplus M \twoheadrightarrow R.$$

Note: $(r, m)(r', m') = (rr', rm' + mr')$.

Quillen cohomology

Definition

Quillen cohomology of X with coefficients in a module M is:

$$\mathrm{HQ}^*(X; M) := \pi^* \operatorname{Hom}(C_\bullet, M)$$

where $C_\bullet \xrightarrow{\sim} X$ is a cofibrant replacement in $s\mathcal{C}$, the category of simplicial objects in $\mathcal{C}$.

Example

For $\mathcal{C} =$ Commutative rings, this is the classic André-Quillen cohomology.

Truncated Π-algebras

Definition

A Π-algebra A is n**-truncated** if it satisfies $A_i = *$ for all $i > n$.

- Postnikov truncation $P_n \colon \mathbf{\Pi Alg} \to \mathbf{\Pi Alg}_1^n$.
- P_n is left adjoint to inclusion $\iota \colon \mathbf{\Pi Alg}_1^n \to \mathbf{\Pi Alg}$.
- Unit map $\eta_A \colon A \to P_n A$.

Truncation Isomorphism

Theorem (F.)

Let A be a Π-algebra and N a module over A which is n-truncated. Then the natural comparison map

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}_1^n}(P_n A; N) \xrightarrow{\cong} \mathrm{HQ}^*_{\mathbf{\Pi Alg}}(A; N).$$

induced by the Postnikov truncation functor P_n is an isomorphism.

Highly connected Π-algebras

Definition

A Π-algebra A is ***n*-connected** if it satisfies $A_i = *$ for all $i \leq n$.

- n-connected cover $C_n \colon \mathbf{\Pi Alg} \to \mathbf{\Pi Alg}^{\infty}_{n+1}$.
- C_n is *right* adjoint to inclusion $\iota \colon \mathbf{\Pi Alg}^{\infty}_{n+1} \to \mathbf{\Pi Alg}$.
- Counit map $\epsilon_A \colon C_n A \to A$.

Connected Cover Isomorphism

Theorem (F.)

Let B be an n-connected Π-algebra and M a module over ιB. Then the natural comparison map

$$\mathrm{HQ}^*_{\mathbf{\Pi Alg}}(\iota B; M) \xrightarrow{\cong} \mathrm{HQ}^*_{\mathbf{\Pi Alg}^\infty_{n+1}}(B; C_n M)$$

induced by the connected cover functor C_n is an isomorphism.

Remark

More general comparison theorem for adjunctions $F \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon G$ between algebraic categories.

2-stage Example

- Take $A_i = 0$ for $i \neq 1, n$.
- A is realizable, e.g., Borel construction

$$BA_1(A_n, n) := EA_1 \times_{A_1} K(A_n, n) \to BA_1.$$

Theorem

$$\mathcal{TM}(A) \simeq \mathrm{Map}_{BA_1}\left(BA_1, BA_1(A_n, n+1)\right)_{h\,\mathrm{Aut}(A)}.$$

Upshot

Classification by a k-invariant is promoted to a **moduli** statement: The **moduli space** of realizations is the **mapping space** where the k-invariant lives.

2-stage Example (cont'd)

Corollary

- $\pi_0 \mathcal{T}\mathcal{M}(A) \simeq H^{n+1}(A_1; A_n) / \operatorname{Aut}(A)$
- *For any choice of basepoint in $\mathcal{T}\mathcal{M}(A)$, we have:*

$$\pi_i \mathcal{T}\mathcal{M}(A) \simeq \begin{cases} 0, \ i > n \\ \operatorname{Der}(A_1, A_n), \ i = n \\ H^{n+1-i}(A_1; A_n), \ 2 \leq i < n \end{cases}$$

and $\pi_1 \mathcal{T}\mathcal{M}(A)$ is an extension by $H^n(A_1; A_n)$ of a subgroup of $\operatorname{Aut}(A)$ *corresponding to realizable automorphisms.*

Stable 2-types

- Take $A_i = 0$ for $i \neq n, n+1$, for some $n \geq 2$.
- A is realizable.

Theorem

$\mathcal{T}\mathcal{M}'(A)$ is connected and its homotopy groups are:

$$\pi_i \mathcal{T}\mathcal{M}'(A) \simeq \begin{cases} 0, \ i \geq 3 \\ \mathrm{Hom}_{\mathbb{Z}}(A_n, A_{n+1}), \ i = 2 \\ \mathrm{Ext}_{\mathbb{Z}}(A_n, A_{n+1}), \ i = 1. \end{cases}$$

Stable 2-types (cont'd)

Corollary

$\mathcal{TM}(A) \simeq \mathcal{TM}'(A)_{h\,\mathrm{Aut}(A)}$ is connected; its homotopy groups are:

$$\pi_i \mathcal{TM}(A) \simeq \begin{cases} 0, & i \geq 3 \\ \mathrm{Hom}_{\mathbb{Z}}(A_n, A_{n+1}) & i = 2 \end{cases}$$

and $\pi_1 \mathcal{TM}(A)$ is an extension of $\mathrm{Aut}(A)$ by $\mathrm{Ext}_{\mathbb{Z}}(A_n, A_{n+1})$. In particular, all automorphisms of A are realizable.

Remark

Few higher automorphisms.

Homotopy operation functors

A Π-algebra A concentrated in degrees $n, n+1, \dots, n+k$ can be described inductively by abelian groups and structure maps:

$$\begin{aligned} & A_n \\ \eta_1 \colon & \Gamma^1_n(A_n) \to A_{n+1} \\ \eta_2 \colon & \Gamma^2_n(A_n, \eta_1) \to A_{n+2} \\ & \dots \\ \eta_k \colon & \Gamma^k_n(\pi_n, \eta_1, \dots, \eta_{k-1}) \to A_{n+k}. \end{aligned}$$

Example

$$\Gamma^1_n(A_n) = \begin{cases} \Gamma(A_n) & \text{for } n = 2 \\ A_n \otimes_{\mathbb{Z}} \mathbb{Z}/2 & \text{for } n \geq 3. \end{cases}$$

and $\eta_1 \colon \Gamma^1_n(A_n) \to A_{n+1}$ is precomposition by the Hopf map $\eta \colon S^{n+1} \to S^n$.

2-stage case

A 2-stage Π-algebra A consists of the data

$$A_n$$

$$\eta_k \colon \widetilde{\Gamma_n^k}(A_n) := \Gamma_n^k(A_n, 0, \ldots, 0) \to A_{n+k}.$$

Example

$\widetilde{\Gamma_3^2}(A_3) = \Lambda(A_3) = A_3 \otimes A_3 / (a \otimes a)$, the exterior square, and
$\eta_2 \colon \Lambda(A_3) \to A_5$ encodes the Whitehead product.

2-stage case (cont'd)

Notation

$Q_{k,n} :=$ indecomposables of $\pi_{n+k}(S^n)$
In the stable range $k \leq n-2$, we have $Q_{k,n} = Q_k^S$, where $Q_*^S :=$ indecomposables of the graded ring π_*^S.

Proposition

Assuming $k \neq n-1$, we have

$$\widetilde{\Gamma_n^k}(A_n) = A_n \otimes_{\mathbb{Z}} Q_{k,n}.$$

In particular, in the stable range we have $\widetilde{\Gamma_n^k}(A_n) = A_n \otimes_{\mathbb{Z}} Q_k^S$.

Criterion for realizability

Theorem (Baues,F.)

The 2-stage Π-algebra given by $\eta_k : \widetilde{\Gamma}_n^k(A_n) \to A_{n+k}$ *is realizable if and only if the map* η_k *factors through the map* $\gamma_{K(A_n,n)}$*:*

$$\begin{array}{ccc} & & H_{n+k+1}K(A_n,n) \\ & \overset{\gamma_{K(A_n,n)}}{\nearrow} & \downarrow \\ \widetilde{\Gamma}_n^k(A_n) & \xrightarrow{\eta_k} & A_{n+k}. \end{array}$$

Criterion for realizability (cont'd)

Corollary

Fix $n \geq 2$ and $k \geq 1$. Then an abelian group A_n has the property that "every Π-algebra concentrated in degrees $n, n+k$ with prescribed group A_n is realizable" if and only if the map

$$\gamma_{K(A_n,n)} \colon \widetilde{\Gamma_n^k}(A_n) \to H_{n+k+1} K(A_n, n)$$

is split injective.

Non-realizable example

First few stable homotopy groups of spheres π_*^S and their indecomposables Q_*^S.

k	π_k^S	Q_k^S
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2\,\langle\eta\rangle$	$\mathbb{Z}/2\,\langle\eta\rangle$
2	$\mathbb{Z}/2\,\left\langle\eta^2\right\rangle$	0
3	$\mathbb{Z}/24 \simeq \mathbb{Z}/8\,\langle\nu\rangle \oplus \mathbb{Z}/3\,\langle\alpha\rangle$	$\mathbb{Z}/12 \simeq \mathbb{Z}/4\,\langle\nu\rangle \oplus \mathbb{Z}/3\,\langle\alpha\rangle$
4	0	0
5	0	0
6	$\mathbb{Z}/2\,\left\langle\nu^2\right\rangle$	0

Non-realizable example (cont'd)

Look at stem $k = 3$.

Proposition

Let $n \geq 5$. The (stable) Π-algebra concentrated in degrees $n, n+3$ given by $A_n = \mathbb{Z}$ and $A_{n+3} = \mathbb{Z}/4$ with structure map

$$\eta_3 \colon A_n \otimes_{\mathbb{Z}} Q_3^S \cong \mathbb{Z}/4 \langle \nu \rangle \oplus \mathbb{Z}/3 \langle \alpha \rangle \twoheadrightarrow \mathbb{Z}/4$$

sending ν to 1 is not realizable.

Proof.

$H\mathbb{Z}_4 H\mathbb{Z} \simeq \mathbb{Z}/6$

$\gamma \colon Q_3^S \simeq \mathbb{Z}/4 \langle \nu \rangle \oplus \mathbb{Z}/3 \langle \alpha \rangle \to H\mathbb{Z}_4 H\mathbb{Z}$ sends 2ν to 0. □

Infinite families

Look at Greek letter elements in the stable homotopy groups of spheres π_*^S.

Proposition

Assume $p \geq 3$.

1. *The first alpha element $\alpha_1 \in Q^S_{2(p-1)-1}$ is **not** in the kernel of γ.*
2. *Higher alpha elements $\alpha_i \in Q^S_{2i(p-1)-1}$ for $i > 1$ are in the kernel of γ.*
3. *Generalized alpha elements $\alpha_{i/j} \in Q^S_*$ for $j > 1$ satisfy $p\alpha_{i/j} \neq 0$ but $\gamma(p\alpha_{i/j}) = 0$.*

Proof.

(3) $\alpha_{i/j}$ has order p^j in π_*^S.
The p-torsion in $H\mathbb{Z}_* H\mathbb{Z}$ is all of order p (and not p^2, p^3, etc.). □

Infinite families (cont'd)

Upshot

This provides infinite families of non-realizable 2-stage (stable) Π-algebras.

Goerss-Hopkins obstruction theory (2004)

- Let E be a homotopy commutative ring spectrum.
- X an E_∞ ring spectrum $\rightsquigarrow$ $E_* X$ is an E_*-algebra in $E_* E$-comodules.
- Realizations of $E_* E$ correspond to E_∞ ring structures on E.
- Applications to chromatic homotopy theory. Morava E-theory E_n admits a unique E_∞ ring structure.

Steenrod problem and variations

- Realizing unstable algebras over the Steenrod algebra as $H^*(X; \mathbb{F}_p)$ for some space X.
- Realizing unstable coalgebras over the Steenrod algebra as $H_*(X; \mathbb{F}_p)$ for some space X. [Blanc (2001), Biedermann–Raptis–Stelzer (2014)]
- Stable analogues.

Power operations

- Let E be an H_∞ ring spectrum.
- X an H_∞ E-algebra $\rightsquigarrow \pi_* X$ is an E_*-algebra with power operations.
- $E = H\mathbb{F}_p$: Dyer-Lashof operations, e.g., acting on the mod p homology of an infinite loop space.
- $E = K_p^\wedge$: θ-algebras over the p-adic integers $\mathbb{Z}_p$.
- $E =$ Morava E-theory E_n: power operations have been studied.

Higher order operations

X a space or spectrum $\rightsquigarrow H^*(X;\mathbb{F}_p)$ a module over the Steenrod algebra (primary cohomology operations)

+ secondary operations

+ tertiary operations

+ etc.

With all higher order cohomology operations, we can recover the p-type of X.

Thank you!