\textbf{A-INFINITY STRUCTURE ON EXT-ALGEBRAS}

\textbf{Abstract.} We give an introduction to $A$-infinity algebras in these notes, which is a generalisation of differential graded algebras. We show that for a graded algebra $A$, the Ext-algebra $\text{Ext}^*_A(k_A, k_A)$ has an $A$-infinity structure that contains sufficient information to recover $A$. On the other hand, we will present an example where the usual associative algebra structure on $\text{Ext}^*_A(k_A, k_A)$ cannot recover $A$. We also show that the $A$-infinity structure is closely related to Massey products.

0.1. Differential graded algebras

We begin by reviewing the definition of a differential graded algebra. Throughout the notes, we use $k$ to denote the ground field unless otherwise stated.

\textbf{Definition 0.1.} A differential graded algebra (in short DG algebra) $A$ over a commutative ring $k$ is a $\mathbb{Z}$-graded $k$-algebra $A = \bigoplus_{p \in \mathbb{Z}} A^p$ together with a differential $d$ of degree 1 such that
\[ d(ab) = (da)b + (-1)^p a(db) \]
for all $a \in A^p$ and $b \in A$. In particular, $A$ is a complex of $k$-modules with differentials $d^n : A^n \to A^{n+1}$, and the cohomology ring $HA$ of a DG $k$-algebra $A$ is a graded associative ring over $k$ with
\[ HA^n = \ker(d^n)/\text{im}(d^{n+1}). \]

\textbf{Example 0.2} (Ext-algebra as the cohomology of a DG algebra). Let $A$ be a connected graded associative algebra over $k$, and let $k_A$ be the trivial $A$-module concentrated in degree 0. The Ext-algebra $\text{Ext}^*_A(k_A, k_A)$ is the cohomology ring of $\text{End}_A(P)$, where $P$ is a free $A$-resolution of $k_A$. $\text{End}_A(P)$ is a DG algebra with
\[ \text{End}_A(P)_p = \prod_{n \in \mathbb{Z}} \text{Hom}_A(P_n, P_{n+p}) \]
and differential $d$ given by
\[ d_p(f) = f \partial + (-1)^{p+1} \partial f, \]
with $f \in \text{End}_A(P)_p$ being a map of degree $p$.

0.2. Recovering the associative algebra from the Ext-algebra

For a connected graded associative algebra $A$ over $k$, we have seen that the classical Ext-algebra $\text{Ext}^*_A(k_A, k_A)$ is the cohomology ring of the DG algebra $\text{End}_A(P)$. Our question is to recover the algebra $A$ from $\text{Ext}^*_A(k_A, k_A)$. Consider the following example:

\textbf{Example 0.3.} Let $A = k\langle x_1, x_2 \rangle /(f)$, with $f = x_1 x_2 + x_2 x_1$ in degree 2. One can show that the minimal free resolution of $k_A$ has the form
\[ \cdots \to 0 \to Ar \to Ae_1 \oplus Ae_2 \to A \to k \to 0, \]
with $e_i$ maps to $x_i$ and $r$ maps to the relation, and
\[ \text{Ext}^*_A(k_A, k_A) = \begin{cases} k & s = 0, \\ k(-1) \oplus k(-1) & s = 1, \\ k(-2) & s = 2, \\ 0 & \text{else}. \end{cases} \]
Write \( E = \text{Ext}^*_A(k_A, k_A) \). In general, we know that \( E^1 \) is dual to \( A_1 \) and \( E^2 \) is dual to the relation \( R = \langle f \rangle = \oplus_{n \geq 2} R_n \) in \( A \). Moreover, restricting the multiplication on \( E \) to \( E^1 \otimes E^1 \), we get a map

\[
E^1 \otimes E^1 \to E^2
\]

that is dual to the inclusion \( R_2 \to A_1 \otimes A_1 \). In this sense, we can recover \( A \) from the Ext-algebra \( E \). See [1] Section 6 for more details of the example.

Before we sketch a proof of the theorem, let us review the example.

By “lifting the identity map”, we mean that there is also a projection map \( p : A \to HA \) that induces a quasi-isomorphism. We will see in the proof that we have choose the projection \( p \) as a vector space splitting. Then the section map \( HA \to A \) will respect the chosen projection \( p \). The maps are not canonically defined.

0.3. \( A \)-infinity algebras

Definition 0.4. An \( A \)-infinity algebra over a base field \( k \) is a \( \mathbb{Z} \)-graded vector space

\[
A = \bigoplus_{p \in \mathbb{Z}} A^p
\]

together with a family of graded \( k \)-linear maps

\[
m_n : A^\otimes n \to A,
\]

of degree \( 2 - n \) for \( n \geq 1 \), satisfying the Stasheff identities \( \text{SI}(n) \):

\[
\sum (-1)^{r+s+t} m_u(id \otimes r \otimes m_s \otimes id \otimes t) = 0,
\]

where the sum runs over all decompositions \( n = r + s + t \), and \( u = r + 1 + t \).

For \( n \) small, the identities have the form:

\begin{itemize}
  \item \( \text{SI}(1) \) \( m_1 m_1 = 0 \);
  \item \( \text{SI}(2) \) \( m_1 m_2 = m_2 m_1 + m_1 \otimes id + id \otimes m_1 \);
  \item \( \text{SI}(3) \) \( m_2(id \otimes m_2 - m_2 \otimes id) = m_1 m_3 + m_3(m_1 \otimes id \otimes id + id \otimes m_1 \otimes id + id \otimes id \otimes m_1) \).
\end{itemize}

In particular, a DG algebra is an \( A \)-infinity algebra with \( m_1 = d \) and \( m_2 = m \) and \( m_n = 0 \) for \( n > 2 \).

We can have a grading on the spaces \( A^p \) too, with

\[
A^p = \bigoplus_{r \in G} A^p_r
\]

indexed by an abelian group \( G \). This grading \( i \) is called the Adams grading, and is denoted by a lower index. The structure maps \( m_n \) are required to respect the Adams grading.

In our examples, we always have \( G = \mathbb{Z} \). In this case, we say that the \( A \)-infinity algebra \( A \) is

Adams connected

if \( A_0 = k \), and \( A = \oplus_{n \geq 0} A_n \) or \( A = \oplus_{n \leq 0} A_n \).

A morphism of \( A \)-infinity algebras consists of a family of \( k \)-linear graded maps

\[
f_n : A^\otimes n \to B
\]

satisfying the Stasheff morphism identities. A morphism \( f \) is a quasi-isomorphism if \( f_1 \) is a quasi-isomorphism.

Theorem 0.5. Let \( A \) be an \( A \)-infinity algebra, and let \( HA \) be the cohomology ring of \( A \). Then there is an \( A \)-infinity structure on \( HA \) with \( m_1 = 0 \) and \( m_2 \) induced by the multiplication on \( A \). And there is a quasi-isomorphism \( HA \to A \) lifting the identity map of \( HA \). This \( A \)-infinity structure on \( HA \) is unique up to quasi-isomorphism.

By “lifting the identity map”, we mean that there is also a projection map \( p : A \to HA \) that induces a quasi-isomorphism. We will see in the proof that we have choose the projection \( p \) as a vector space splitting. Then the section map \( HA \to A \) will respect the chosen projection \( p \). The maps are not canonically defined.

Before we sketch a proof of the theorem, let us review the example.
Example 0.6. For $A = k[x_1, x_2]/(f)$ with $f \in A^q$. We see that the only non-zero multiplication on $E$ is $m_q$. And one can show that the restriction of $m_q$ to $(E^1)^{\otimes q}$ is dual to the inclusion $R_n \to A^{\otimes n}$. The result is made more general in [1, Theorem A] for $\sum_{s+t=n,s,t \geq 1} \lambda_2(\lambda_s \otimes \lambda_t)$, assuming that $\lambda_s$ and $\lambda_t$ are defined for smaller $s$ and $t$. One immediately sees that there are two problems here: the cohomology plays no role in this formula, and the degrees do not match. The right formula that will fix the problems is as follows:

$$\lambda_n = \sum_{s+t=n,s,t \geq 1} \lambda_2(G \lambda_s \otimes G \lambda_t),$$

where $G$ is a homotopy on $A$ from the identity map $id_A$ to the projection $p$ onto $HA$. Here we identify $HA$ with $\oplus H^n$ as a subspace of $A$, and choose a splitting $A^n = B^n \oplus H^n \oplus L^n$, and $p$ is the projection onto the summand $H \subseteq A$. Since $L^{n-1} \cong B^n$, we can choose $G$ to respect the splitting: $G|B^n \cong L^{n-1} \subseteq A^{n-1}$ and $G|H^{n} \oplus L^n = 0$. For $n = 1$, we set $G \lambda_1$ formally to be the identity map. Now one can check that the maps

$$p(\lambda_n|HA) : HA \to HA$$

endows $HA$ with an $A$-infinity structure.

0.4. $A$-infinity algebras and Massey products

Let $A$ be a DG algebra. Let $\alpha_1$, $\alpha_2$, and $\alpha_3$ be classes in $HA$ represented by $a_{01}$, $a_{12}$, and $a_{23}$ in $A$. Assume that $\alpha_1 \alpha_2 = \alpha_2 \alpha_3 = 0$. Set $a_{02} = G(a_{01}a_{12})$ and $a_{13} = G(a_{12}a_{23})$. Then $\partial(a_{02}) = a_{01}a_{12}$ and $\partial(a_{13}) = a_{12}a_{23}$. Up to signs, this is what we need to define the three-fold Massey product $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in HA$, so $(-1)^b m_n(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. In general, this fact holds for higher products too.

Theorem 0.7. Let $A$ be a DG algebra. Let $\alpha_1, \ldots, \alpha_n$ be classes in $HA$ such that the $n$-fold Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined. Then

$$(-1)^b m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in (a_1, \ldots, a_n),$$

where $b = 1 + \deg(\alpha_{n-1}) + \deg(\alpha_{n-3}) + \deg(\alpha_{n-5}) + \cdots$.

Remark 0.8. Recall that the homotopy $G : A \to A$ depends on a splitting of $A$, so we can have different homotopies $G$ that produce different classes in the Massey product, but this process does not necessarily produce all the classes in the Massey product.

Example 0.9. Let $p$ be an odd prime, and let $k$ be a field of characteristic $p$. Take $A = k[x]/(x^p)$ with $x$ in Adams degree 2d. Then the Ext-algebra of $A$ is

$$\text{Ext}^*_A(k_A, k_A) \cong \Lambda(y_1) \otimes k[y_2],$$

with $y_1$ in degree $(1, -2d)$ and $y_2$ in degree $(2, -2dp)$. Moreover, we have $m_p(y_1 \otimes \cdots \otimes y_1) = y_2$, and one can compute that the $p$-fold Massey product $\langle y_1, \ldots, y_1 \rangle = \{(-1)^{(p+1)/2}y_2\}.$

References