Math 241 - Calculus III
Spring 2012, section CL1
§ 14.5. Chain rule

1 Functions of 2 variables

Consider a function of 2 variables $f(x, y)$, e.g. the temperature in a room. Let's say x and y are themselves functions of some variable t, e.g. $(x(t), y(t))$ is a parametrized curve representing the position of a particle at time t. We are interested in the temperature of the particle and how it changes with time. In other words, we are interested in the function $f(x(t), y(t))$ and its derivative.

The chain rule says:

$$\frac{d}{dt}f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt}. \tag{1}$$

In slightly more compact notation:

$$\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t). \tag{2}$$

Let us rewrite the chain rule using notation that is less rigorous but easier to read and to remember:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}. \tag{3}$$

Warning: When using this convenient but ambiguous notation (3), please remember where each quantity must be evaluated, as specified in the variants (1) and (2).

Example 1. The curve is a circle of radius $\sqrt{2}$ going counterclockwise around the origin:

$$\begin{cases} x(t) = \sqrt{2}\cos t \\ y(t) = \sqrt{2}\sin t \end{cases}$$

(as in figure 1) and the function is $f(x, y) = (x + y^2)^2$. Let $h(t) := f(x(t), y(t))$. Find $h'(\frac{\pi}{4})$.

Solution. At time $t = \frac{\pi}{4}$, the particle is at position

$$\begin{cases} x(\frac{\pi}{4}) = \sqrt{2}\cos\frac{\pi}{4} = 1 \\ y(\frac{\pi}{4}) = \sqrt{2}\sin\frac{\pi}{4} = 1 \end{cases}$$

while its velocity is

$$\begin{cases} x'(t) = -\sqrt{2}\sin t \\ x'(\frac{\pi}{4}) = -\sqrt{2}\sin\frac{\pi}{4} = -1 \\ y'(t) = \sqrt{2}\cos t \\ y'(\frac{\pi}{4}) = \sqrt{2}\cos\frac{\pi}{4} = 1. \end{cases}$$

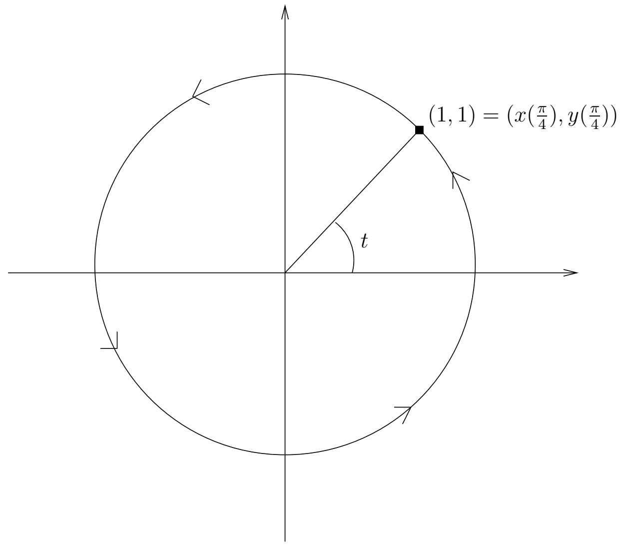

Figure 1: Circle around the origin.

The partial derivatives of f are

$$
\begin{aligned}
f_x &= 2(x+y^2) \\
f_y &= 2(x+y^2)(2y) = 4y(x+y^2).
\end{aligned}
$$

The chain rule gives us

$$
\begin{aligned}
h'(\frac{\pi}{4}) &= f_x(x(\frac{\pi}{4}), y(\frac{\pi}{4}))x'(\frac{\pi}{4}) + f_y(x(\frac{\pi}{4}), y(\frac{\pi}{4}))y'(\frac{\pi}{4}) \\
&= f_x(1,1)(-1) + f_y(1,1)(1) \\
&= 4(-1) + 8(1) = 4. \square
\end{aligned}
$$

Remark: We did not really need the chain rule in this case. We can explicitly write down the function

$$
\begin{aligned}
h(t) &= f(x(t), y(t)) \\
&= (x(t) + y(t)^2) \\
&= (\sqrt{2}\cos t + 2\sin^2 t)^2
\end{aligned}
$$

then compute its derivative

$$
h'(t) = 2(\sqrt{2}\cos t + 2\sin^2 t)(-\sqrt{2}\sin t + 4\sin t\cos t)
$$

and evaluate at $t = \frac{\pi}{4}$ to find

$$
\begin{aligned}
h'(\frac{\pi}{4}) &= 2(\sqrt{2}\cos\frac{\pi}{4} + 2\sin^2\frac{\pi}{4})(-\sqrt{2}\sin\frac{\pi}{4} + 4\sin\frac{\pi}{4}\cos\frac{\pi}{4}) \\
&= 2(1+1)(-1+2) \\
&= 4. \square
\end{aligned}
$$

Question: How is the chain rule useful if we can do without it?

Answer: We don't always know what the function f is. In real life, it could be a function estimated from a few sample data points.

Example 2. The curve is a circle as in Example 1 but this time, the function $f(x, y)$ is *unknown*. All we know is the value of the partial derivatives

$$f_x(1, 1) = 17$$
$$f_y(1, 1) = 30.$$

Let $h(t) := f(x(t), y(t))$. Find $h'(\frac{\pi}{4})$.

Solution. Although we cannot describe the function $h(t)$ explicitly, the chain rule gives us

$$\begin{aligned} h'(\frac{\pi}{4}) &= f_x(x(\frac{\pi}{4}), y(\frac{\pi}{4}))x'(\frac{\pi}{4}) + f_y(x(\frac{\pi}{4}), y(\frac{\pi}{4}))y'(\frac{\pi}{4}) \\ &= f_x(1, 1)(-1) + f_y(1, 1)(1) \\ &= 17(-1) + 30(1) = 13. \square \end{aligned}$$

2 Sketch of proof

Here is a heuristic argument to prove the chain rule. Consider only linear approximations and neglect all higher order error terms. If time t increases by a very small amount Δt, then the position $(x(t), y(t))$ of the particle changes by amounts

$$\begin{cases} \Delta x \approx \frac{dx}{dt}\Delta t \\ \Delta y \approx \frac{dy}{dt}\Delta t. \end{cases}$$

Therefore the function f changes by the amount

$$\begin{aligned} \Delta f &\approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \\ &\approx \frac{\partial f}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial f}{\partial y}\frac{dy}{dt}\Delta t \end{aligned}$$

so that the rate of change is approximately

$$\frac{\Delta f}{\Delta t} \approx \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

and in fact the instantaneous rate of change is indeed

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

With more care, this heuristic argument can be made rigorous.

3 Functions of many variables

Consider a function of 3 variables $f(x, y, z)$, all of which are themselves functions $x(t), y(t), z(t)$ of a variable t. This could describe the temperature of a particle moving in 3-space.

Using the same shorthand notation as in (3), the chain rule says:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}}. \tag{4}$$

The pattern is the same for functions of any number of variables.

Example 3: Consider $f(x, y, z) = x^2 + yz$ and

$$\begin{cases} x(t) = t \\ y(t) = t^2 \\ z(t) = 1 - t. \end{cases}$$

Let $h(t) := f(x(t), y(t), z(t))$. Find $h'(t)$.

Solution. The chain rule gives us

$$\begin{aligned} \frac{dh}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= (2x)(1) + (z)(2t) + (y)(-1) \\ &= (2t)(1) + (1-t)(2t) + (t^2)(-1) \\ &= 2t + 2t - 2t^2 - t^2 \\ &= 4t - 3t^2. \ \square \end{aligned}$$

Remark: Here again, we did not need the chain rule, because we know the functions f, x, y, and z explicitly. We can write down the function

$$\begin{aligned} h(t) &= f(x(t), y(t), z(t)) \\ &= x(t)^2 + y(t)z(t) \\ &= t^2 + t^2(1-t) \\ &= 2t^2 - t^3 \end{aligned}$$

and compute its derivative

$$h'(t) = 4t - 3t^2. \ \square$$

4 Several independent variables

Consider $f(x, y)$ where x and y are themselves functions $x(s, t)$ and $y(s, t)$ of 2 independent variables s and t. We are interested in the function $f(x(s, t), y(s, t))$ and its partial derivatives with respect to s and t.

Because partial derivatives are computed by treating the other variables as constants, the chain rule yields

$$\boxed{\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}} \tag{5}$$

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}. \tag{6}$$

Example 4: Consider

$$\begin{cases} x(s, t) = s^2 + 5t \\ y(s, t) = 3s - t^2 \end{cases}$$

and the function $f(x, y) = e^{xy}$. Let $h(s, t) := f(x(s, t), y(s, t))$. Find $h_s(2, 1)$ and $h_t(2, 1)$.

Solution. We compute

$$\begin{cases} x(2, 1) = 4 + 5 = 9 \\ y(2, 1) = 6 - 1 = 5 \end{cases}$$

and the partial derivatives

$$\begin{cases} x_s = 2s \\ x_s(2, 1) = 4 \\ x_t = 5 \\ x_t(2, 1) = 5 \\ y_s = 3 \\ y_s(2, 1) = 3 \\ y_t = -2t \\ y_t(2, 1) = -2. \end{cases}$$

The partial derivatives of f are

$$f_x = ye^{xy}$$
$$f_y = xe^{xy}.$$

The chain rule gives us

$$\begin{aligned} h_s(2, 1) &= f_x(x(2, 1), y(2, 1))x_s(2, 1) + f_y(x(2, 1), y(2, 1))y_s(2, 1) \\ &= f_x(9, 5)(4) + f_y(9, 5)(3) \\ &= 5e^{45}(4) + 9e^{45}(3) \\ &= (20 + 27)e^{45} \\ &= 47e^{45} \end{aligned}$$

$$
\begin{aligned}
h_t(2,1) &= f_x(x(2,1),y(2,1))x_t(2,1) + f_y(x(2,1),y(2,1))y_t(2,1) \\
&= f_x(9,5)(5) + f_y(9,5)(-2) \\
&= 5e^{45}(5) + 9e^{45}(-2) \\
&= (25-18)e^{45} \\
&= 7e^{45}. \ \square
\end{aligned}
$$

Remark: Here again, we did not need the chain rule, because we know the functions f, x, and y explicitly.

Example 5: As in example 4, consider

$$
\begin{cases} x(s,t) = s^2 + 5t \\ y(s,t) = 3s - t^2 \end{cases}
$$

and some unknown function $f(x,y)$ with partial derivatives

$$
\begin{aligned}
f_x(9,5) &= 7 \\
f_y(9,5) &= -3.
\end{aligned}
$$

Let $h(s,t) := f(x(s,t),y(s,t))$. Find $h_s(2,1)$ and $h_t(2,1)$.

Solution. Although we cannot describe the function $h(s,t)$ explicitly, the chain rule gives us

$$
\begin{aligned}
h_s(2,1) &= f_x(x(2,1),y(2,1))x_s(2,1) + f_y(x(2,1),y(2,1))y_s(2,1) \\
&= 7(4) + (-3)(3) \\
&= 28 - 9 \\
&= 19
\end{aligned}
$$

$$
\begin{aligned}
h_t(2,1) &= f_x(x(2,1),y(2,1))x_t(2,1) + f_y(x(2,1),y(2,1))y_t(2,1) \\
&= f_x(9,5)(5) + f_y(9,5)(-2) \\
&= 7(5) + (-3)(-2) \\
&= 35 + 6 \\
&= 41. \ \square
\end{aligned}
$$