Math 241 - Calculus III
Spring 2012, section CL1
§ 16.7. Surface integrals of vector fields

In these notes, we illustrate how to compute the flux integral of a vector field $\vec{F}$ across a surface S. If S is parametrized by $\vec{r}(u,v)$ with domain of parametrization D, then the flux integral is computed by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

up to a sign, **depending on the orientation** of the surface. Really we should write $\pm \vec{r}_u \times \vec{r}_v$.

Let us solve an example from the book.

Problem 16.7.22. The vector field is $\vec{F}(x,y,z) = (x, y, z^4)$ and the surface S is the part of the cone $z = \sqrt{x^2 + y^2}$ beneath the plane $z = 1$ with downward orientation.

Solution. Let us parametrize S using z and θ:

$$\vec{r}(z,\theta) = (z\cos\theta, z\sin\theta, z)$$

with domain $0 \leq z \leq 1$, $0 \leq \theta \leq 2\pi$. The tangent vectors are

$$\vec{r}_z = (\cos\theta, \sin\theta, 1)$$
$$\vec{r}_\theta = (-z\sin\theta, z\cos\theta, 0)$$

so that a normal vector is

$$\begin{aligned}\vec{r}_z \times \vec{r}_\theta &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -z\sin\theta & z\cos\theta & 0 \end{vmatrix} \\ &= \vec{i}(-z\cos\theta) - \vec{j}(z\sin\theta) + \vec{k}(z\cos^2\theta + z\sin^2\theta) \\ &= (-z\cos\theta, -z\sin\theta, z).\end{aligned}$$

Because this normal vector points up, whereas we want the **downward orientation** of S, we take instead

$$\vec{r}_\theta \times \vec{r}_z = (z\cos\theta, z\sin\theta, -z).$$

The flux across S is

$$
\begin{aligned}
\iint_S \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^1 \vec{F} \cdot (z\cos\theta, z\sin\theta, -z) dz d\theta \\
&= \int_0^{2\pi} \int_0^1 (x)(z\cos\theta) + (y)(z\sin\theta) + (z^4)(-z) dz d\theta \\
&= \int_0^{2\pi} \int_0^1 (z\cos\theta)(z\cos\theta) + (z\sin\theta)(z\sin\theta) + (z^4)(-z) dz d\theta \\
&= \int_0^{2\pi} \int_0^1 z^2 - z^5 dz d\theta \\
&= \int_0^{2\pi} \left[\frac{z^3}{3} - \frac{z^6}{6}\right]_0^1 d\theta \\
&= \left(\frac{1}{3} - \frac{1}{6}\right) \int_0^{2\pi} d\theta \\
&= \frac{1}{6}(2\pi) \\
&= \boxed{\frac{\pi}{3}}.
\end{aligned}
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