Thursday, March 15 $\quad **$ *Worksheet 15 - Introduction to multiple integrals*

1. Evaluate the following integral by reversing the order of integration:

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy.$$

   (Hint: When you change to $dx\, dy$, be sure to also change the bounds of integration.)

2. Consider the region bounded by the curves determined by $-2x + y^2 = 6$ and $-x + y = -1$.

   (a) Sketch the region R in the plane.

   (b) Set up and evaluate an integral of the form $\iint_R dA$ that calculates the area of R.

3. Consider the region R which lies above the x-axis and between the circles of radius 1 and 2 centered at $(0,0)$. *Without using polar coordinates*, evaluate

$$\iint_R y\, dA.$$

4. Evaluate

$$\int_{-2}^0 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\, dy\, dx.$$

   Hint: don't do it directly.

5. The function $P(x) = e^{-x^2}$ is fundamental in probability.

   (a) Sketch the graph of $P(x)$. Explain why it is called a "bell curve."

   (b) Compute $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$ using the following brilliant strategy of Gauss.

      i. Instead of computing I, compute $I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right)$.

      ii. Rewrite I^2 as an integral of the form $\iint_R f(x,y)\, dA$ where R is the entire Cartesian plane.

      iii. Convert that integral to polar coordinates.

      iv. Evaluate to find I^2. Deduce the value of I.

   Amazingly, it can be mathematically proven that there is NO elementary function $Q(x)$ (that is, function built up from sines, cosines, exponentials, and roots using "usual" operations) for which $Q'(x) = P(x)$.

6. Compute $\displaystyle\int_0^{\infty} \int_0^{\infty} \frac{1}{(1+x^2+y^2)^2}\, dx\, dy$.