**Math 285 - Intro Differential Equations**
**Spring 2011, sections G1 and X1**
**Deflection of a beam**

## Section 3.8

**16a.** The shape of the beam is given by:

$$y(x) = \frac{w}{24EI}x^4 + Ax^3 + Bx^2 + Cx + D$$

where $A$, $B$, $C$, $D$ are coefficients to be determined from the endpoint conditions. Since the beam is fixed at its ends $x = 0$ and $x = L$, the endpoint conditions are:

$$y(0) = y'(0) = 0$$
$$y(L) = y'(L) = 0.$$

The conditions at $x = 0$ yield:

$$y(0) = D = 0$$
$$y'(0) = C = 0.$$

The conditions at $x = L$ then yield:

$$\begin{aligned} y(L) &= \frac{w}{24EI}L^4 + AL^3 + BL^2 = 0 \\ &\Rightarrow \frac{w}{24EI}L^2 + AL + B = 0 \qquad (1) \\ y'(L) &= \frac{w}{6EI}L^3 + 3AL^2 + 2BL = 0 \\ &\Rightarrow \frac{w}{6EI}L^2 + 3AL + 2B = 0. \qquad (2) \end{aligned}$$

Subtracting respectively 2 or 3 times equation (1) from equation (2) yields:

$$\frac{w}{12EI}L^2 + AL = 0$$
$$\Rightarrow A = -\frac{w}{12EI}L$$
$$\frac{w}{24EI}L^2 - B = 0$$
$$\Rightarrow B = \frac{w}{24EI}L^2$$

so that the shape of the beam is given by:

$$\begin{aligned} y(x) &= \frac{w}{24EI}x^4 - \frac{w}{12EI}Lx^3 + \frac{w}{24EI}L^2x^2 \\ &= \frac{w}{24EI}\left(x^4 - 2Lx^3 + L^2x^2\right). \end{aligned}$$

**b.** The roots of $y'(x) = 0$ are (ignoring the constant factor) the roots of:

$$4x^3 - 6Lx^2 + 2L^2x = 0$$
$$x(2x^2 - 3Lx + L^2) = 0$$
$$x(2x - L)(x - L) = 0$$

that is $x = 0, \frac{L}{2}, L$. Since the highest-degree term of $y(x)$ is $\frac{w}{24EI}x^4$, the three critical points of $y$ are respectively a local minimum, a local maximum, and a local minimum. The maximum of $y$ on the interval $[0, L]$ is therefore attained either at $x = \frac{L}{2}$ or at the endpoints, which is not the case. So the maximum is:

$$\begin{aligned}
y_{\max} &= y(\frac{L}{2}) \\
&= \frac{w}{24EI}\left(\frac{L}{2}\right)^2\left((\frac{L}{2})^2 - 2L(\frac{L}{2}) + L^2\right) \\
&= \frac{w}{24EI}\frac{L^2}{4}\left(\frac{L^2}{4} - L^2 + L^2\right) \\
&= \frac{w}{24EI}\frac{L^4}{16} \\
&= \frac{wL^4}{384EI}.
\end{aligned}$$