Math 285 - Intro Differential Equations
Spring 2011, sections G1 and X1
Computing Fourier coefficients

Section 9.1

17. $f(t) = |t|, \ -\pi \le t \le \pi$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos nt\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} t\cos nt\,dt \ \text{ since } |t|\cos nt \text{ is even} \\
u &= t \qquad dv = \cos nt\,dt \\
du &= dt \qquad v = \frac{1}{n}\sin nt \ \text{ assuming } n \ge 1 \\
&= \frac{2}{\pi}\left[\frac{1}{n}t\sin nt|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nt\,dt\right] \\
&= \frac{2}{\pi n}\left[\frac{1}{n}\cos nt\right]_0^{\pi} \\
&= \frac{2}{\pi n^2}\left((-1)^n - 1\right) \\
&= \begin{cases} -\frac{4}{\pi n^2} \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \\
&= \frac{2}{\pi}\int_0^{\pi} t\,dt \\
&= \frac{2}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} \\
&= \pi.
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \\
&= 0 \text{ since } f(t)\sin nt \text{ is odd.}
\end{aligned}$$

The Fourier series of f is:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$$
$$=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos t+\frac{1}{9}\cos 3t+\frac{1}{25}\cos 5t+\frac{1}{49}\cos 7t+\dots\right)$$

Remark: If we love computing integrals, we can compute the coefficients b_n explicitly:

$$b_n=\frac{1}{\pi}\left(\int_{-\pi}^{0}(-t)\sin nt\,dt+\int_0^{\pi}t\sin nt\,dt\right)$$

$$\int_{-\pi}^{0}t\sin nt\,dt$$

$$u=t \qquad dv=\sin nt\,dt$$

$$du=dt \qquad v=-\frac{1}{n}\cos nt$$

$$=-\frac{1}{n}\left[(t\cos nt)_{-\pi}^{0}-\int_{-\pi}^{0}\cos nt\,dt\right]$$
$$=-\frac{1}{n}(0-(-\pi)(-1)^n)$$
$$=\frac{-\pi(-1)^n}{n}$$

$$\int_0^{\pi}t\sin nt\,dt$$
$$=-\frac{1}{n}\left[(t\cos nt)_0^{\pi}-\int_0^{\pi}\cos nt\,dt\right]$$
$$=-\frac{1}{n}(\pi(-1)^n-0)$$
$$=\frac{-\pi(-1)^n}{n}$$

$$b_n=\frac{1}{\pi}\left(-\int_{-\pi}^{0}t\sin nt\,dt+\int_0^{\pi}t\sin nt\,dt\right)$$
$$=\frac{1}{\pi}\left(\frac{\pi(-1)^n}{n}-\frac{\pi(-1)^n}{n}\right)$$
$$=0.$$