Math 415 - Applied Linear Algebra
Fall 2010, section D1
Practice Final, Monday December 6

Name: ____

This is a practice exam to prepare for the final. It is longer and more difficult than the actual exam.

Problem 1a. Find all solutions (if any) of the system:

$$\begin{aligned} 3x_1 - x_2 + 5x_3 &= 0 \\ 5x_1 + x_2 + 9x_3 &= 2 \\ -x_1 + 11x_2 + x_3 &= 8 \\ x_1 + x_2 + 2x_3 &= 1. \end{aligned}$$

b. Is the vector $b = \begin{bmatrix} 0 \\ 2 \\ 8 \\ 1 \end{bmatrix}$ in Span$\{\begin{bmatrix} 3 \\ 5 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 11 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 1 \\ 2 \end{bmatrix}\}$? If not, explain; if so, **write** b **explicitly** as a linear combination of the given vectors.

c. What is the dimension of Span$\{\begin{bmatrix} 3 \\ 5 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 11 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 1 \\ 2 \end{bmatrix}\}$? Explain.

Problem 2a. Find **all values** of α such that the matrix $A = \begin{bmatrix} 0 & 1 & \alpha \\ 1 & 0 & 1 \\ 1 & \alpha & 5 \end{bmatrix}$ fails to be invertible.

b. For $\alpha = 0$, find the inverse of A.

c. Again for $\alpha = 0$, find all solutions (if any) of the equation $Ax = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

Problem 3. Let A be an $m \times n$ matrix of rank r. In each case below, answer the following.

(i) Give some example of what A could be.

(ii) Is the system $Ax = b$ consistent for every $b \in \mathbb{R}^m$? Explain.

(iii) When the system $Ax = b$ has a solution, is the solution unique? Explain.

a. $m = 3$, $n = 2$, $r = 2$

(i)

(ii)

(iii)

b. $m = 2$, $n = 3$, $r = 2$

(i)

(ii)

(iii)

c. $m = 3$, $n = 3$, $r = 2$

(i)

(ii)

(iii)

d. $m = 3$, $n = 3$, $r = 3$

(i)

(ii)

(iii)

Problem 4. Let $f_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $f_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $g_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $g_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

a. Find the transition matrix from the basis $\{f_1, f_2\}$ to the basis $\{g_1, g_2\}$.

b. Find the coordinates of the vector $v = -f_1 + 3f_2$ with respect to the basis $\{g_1, g_2\}$.

Problem 5. Let $P_3 = \{p(x) = c_0 + c_1x + c_2x^2 \mid c_0, c_1, c_2 \in \mathbb{R}\}$ be the vector space of polynomials of degree less than 3. Consider the map $L\colon P_3 \to \mathbb{R}^3$ which evaluates polynomials at three **distinct** points a, b, c in $\mathbb{R}$:

$$L(p) = \begin{bmatrix} p(a) \\ p(b) \\ p(c) \end{bmatrix}.$$

For example, the polynomial $x^2 + 3x - 2$ is sent to the vector $\begin{bmatrix} a^2 + 3a - 2 \\ b^2 + 3b - 2 \\ c^2 + 3c - 2 \end{bmatrix}$.

a. Is L a linear transformation? Prove your answer.

b. Find the matrix representation of L with respect to the monomial basis $\{1, x, x^2\}$ of P_3 and the standard basis of $\mathbb{R}^3$.

c. Find the rank of L, i.e. the dimension of $\mathrm{im}(L)$, and the nullity of L, i.e. the dimension of $\ker(L)$. [Hint: You can work in coordinates. The rank and nullity can be computed using any matrix representation.]

d. Using part (c), find $\mathrm{im}(L)$ and $\ker(L)$.

e. Part (d) says something about prescribing values for polynomials. Say we're looking for a polynomial p of degree less than 3 with prescribed values $p(a) = y_1$, $p(b) = y_2$, $p(c) = y_3$. Can we always find one? When we can, how many are there?

Problem 6. Let A be a 4×3 matrix of rank 2 such that the third column a_3 is equal to $5a_1 - a_2$. Find a basis of the row space of A.

Problem 7. You own shares in a startup company and are trying to determine how their value $f(t)$ changes over time. There seems to be a long term trend going up at a constant rate c, and temporary fluctuations that oscillate with amplitude d. The value could be a function of the form $f(t) = ct + d\sin(\frac{\pi}{2}t)$ for some coefficients c,d.

Find the least squares fit of that form through data points $(1, 2)$, $(2, 4)$, $(3, 5)$, where the first coordinate is the time t and the second coordinate is the observed value.

Problem 8. Consider the space $C[0,1]$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{dx}$. We want to approximate the function x^2 by a polynomial of degree at most one. Find the function $c_0 + c_1 x$ which is closest to x^2 in $C[0,1]$.

Problem 9. Let A be a 4×4 matrix such that the fourth column is the sum of the first three columns. Find an eigenvalue of A and corresponding eigenvector. Explain.

Problem 10. Let A be a 4×4 diagonalizable matrix with a single eigenvalue λ. Find A.

Problem 11. Solve the initial value problem:

$$\begin{aligned}
x_1' &= 2x_1 - 2x_2 \\
x_2' &= 2x_1 - 3x_2 \\
x_1(0) &= 2 \\
x_2(0) &= 7.
\end{aligned}$$

Problem 12a. Find **all values** of α and β such that the matrix $A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ \alpha & \beta & 5 \end{bmatrix}$ can be diagonalized by an orthogonal matrix. Explain.

b. For some values of α and β found in (a), write a diagonalization of A by an orthogonal matrix U.