Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Additional problems

Section 6.1

For the following problems, we recall the setup of Theorem 1.1 in § 6.1 as well as its proof.

Let X be an n-dimensional complex inner product space, and $A\colon X \to X$ a linear map. Let λ_1 be an eigenvalue of A with corresponding eigenvector u_1, i.e. $Au_1 = \lambda_1 u_1$, and assume u_1 has been normalized i.e. $\|u_1\| = 1$.

Denote $E := u_1^{\perp}$, the orthogonal complement of the line $\mathrm{Span}\{u_1\}$ in X. Let $\{v_2, \ldots, v_n\}$ be an **orthonormal** basis of E, so that $\{u_1, v_2, \ldots, v_n\}$ is an **orthonormal** basis of X. In that basis, the matrix of A has the form

$$\left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & \\ \vdots & A_1 \\ 0 & \end{array}\right]. \tag{1}$$

By abuse of notation, we denote by $A_1\colon E \to E$ the map whose matrix in the basis $\{v_2, \ldots v_n\}$ is A_1.

A6.1.1. Show that A_1 is the composite $A_1 = \mathrm{Proj}_E \circ A$, where $\mathrm{Proj}_E\colon X \to E$ denotes the orthogonal projection onto E. In diagrams:

$$E \xrightarrow{A} X \xrightarrow{\mathrm{Proj}_E} E. \qquad (E \xrightarrow{A_1} E)$$

A6.1.2. In general, the subspace E is not invariant under A, that is we have $AE \not\subseteq E$. Equivalently, the upper-right block $*$ of the matrix (1) is not zero. Let us illustrate this by an example.

Consider the linear map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ whose standard matrix is

$$\begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix}$$

a. Carry out the process described above. More explicitly: find appropriate λ_1, u_1, v_2 and find the matrix of A in the basis $\{u_1, v_2\}$ of $\mathbb{R}^2$. The upper-right 1×1 block $*$ should be a non-zero entry.

b. Because A is 2×2, the algorithm stops here and your answer to part (a) is a Schur representation of A. Call it T (for upper **T**riangular). Letting $U = \begin{bmatrix} u_1 & v_2 \end{bmatrix}$, verify explicitly $A = UTU^*$ by computing the matrix multiplication.