Math 416 - Abstract Linear Algebra
Fall 2011, section E1
Practice midterm 1

Name: ____

- This is a practice exam. The real exam will consist of at most 4 problems.
- In the real exam, no calculators, electronic devices, books, or notes may be used.
- Show your work. No credit for answers without justification.
- Good luck!

1. __________/10
2. __________/10
3. __________/10
4. __________/10
5. __________/10
6. __________/10
7. __________/10
8. __________/10

Total: __________/80

Problem 1a. (5 pts) Find all solutions (if any) of the system

$$\begin{cases} x_1 + 3x_2 + 2x_3 & = 2 \\ x_1 + 6x_2 + x_3 & = 3 \\ 2x_1 + 3x_2 + 5x_3 & = 5. \end{cases}$$

b. (5 pts) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 6 & 1 \\ 2 & 3 & 5 \end{bmatrix}$. Find a basis of $\operatorname{Null} A$.

Problem 2. Let $C(\mathbb{R})$ denote the vector space of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. Let $T\colon C(\mathbb{R}) \to \mathbb{R}$ be the transformation defined by

$$T(f) = \int_1^2 f(x)\,\mathrm{dx}.$$

a. (4 pts) Is T linear? Prove your answer.

b. (4 pts) Consider the subspace $P_2 \subset C(\mathbb{R})$ of polynomial functions of degree at most 2. Find the matrix representing $T\colon P_2 \to \mathbb{R}$ with respect to the monomial basis $\{1, x, x^2\}$ of P_2.

c. (2 pts) Compute $T(13 + 8x + 9x^2)$.

Problem 3. (10 pts) Let $p(t)$ be a polynomial and take its derivatives until you reach zero:

$$p', p'', p^{(3)}, \ldots, p^{(k)} \neq 0, p^{(k+1)} = 0.$$

Show that the collection $\{p, p', p'', \ldots, p^{(k)}\}$ is linearly independent (in the space of all polynomials). Hint: Degree.

Problem 4. A 3×3 matrix will be called a **magic square** if the entries of each row and each column add up to 0. For example, here is a magic square:

$$\begin{bmatrix} 2 & -3 & 1 \\ 2 & -2 & 0 \\ -4 & 5 & -1 \end{bmatrix}.$$

a. (4 pts) Show that the set of all magic squares $S := \{A \in M_{3\times 3} \mid A \text{ is magic } \}$ is a subspace of $M_{3\times 3}$.

b. (6 pts) Find a basis of S.

Problem 5. Let A be an $m \times n$ matrix and B an $n \times p$ matrix.

a. (5 pts) If A and B have linearly independent columns, does AB have linearly independent columns? Prove your answer.

b. (5 pts) If the columns of A span $\mathbb{R}^m$ and the columns of B span $\mathbb{R}^n$, do the columns of AB span $\mathbb{R}^m$? Prove your answer.

Problem 6a. (4 pts) Do row operations preserve the column space? In other words, if we have row equivalent matrices $A \sim B$, can we conclude $\operatorname{Col} A = \operatorname{Col} B$? Prove your answer.

b. (4 pts) Do row operations preserve the null space? Prove your answer.

c. (2 pts) Do row operations preserve linear dependence relations among columns? Prove your answer. (A linear dependence relation among vectors $v_1, \ldots, v_n$ is a linear combination adding up to zero: $c_1 v_1 + \ldots + c_n v_n = \vec{0}$.)

Problem 7. Consider the vectors $v_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $v_4 = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$.

a. (5 pts) Among these four vectors, select a basis of $\mathrm{Span}\{v_1, v_2, v_3, v_4\}$.

b. (5 pts) Express each vector that you discarded as a linear combination of the vectors that you kept. Hint: Problem 6c.

Problem 8. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.

a. (6 pts) Is A invertible? If not, justify; if so, find A^{-1}.

b. (4 pts) Find all solutions (if any) of the equation $Ax = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.