Math 527 - Homotopy Theory
Additional notes

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1 Group objects

Definition 1.1. Let $\mathcal{C}$ be a category with finite products, including a terminal object 1. A **group object** in $\mathcal{C}$ is an object G of $\mathcal{C}$ together with structure maps

$$\mu \colon G \times G \to G \quad \text{“multiplication”}$$

$$e \colon 1 \to G \quad \text{“unit”}$$

$$i \colon G \to G \quad \text{“inverse”}$$

such that the following diagrams commute:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id} \times \mu} & G \times G \\ {\scriptstyle \mu \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mu} \\ G \times G & \xrightarrow[\mu]{} & G \end{array} \qquad \text{(Associativity)}$$

$$\begin{array}{ccc} 1 \times G & \xrightarrow{e \times \mathrm{id}} & G \times G \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \mu} \\ & & G \end{array} \qquad \text{(Left unit)}$$

$$\begin{array}{ccc} G \times 1 & \xrightarrow{\mathrm{id} \times e} & G \times G \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \mu} \\ & & G \end{array} \qquad \text{(Right unit)}$$

$$\begin{array}{ccc} G & \xrightarrow{(i, \mathrm{id})} & G \times G \\ & {\scriptstyle e_G} \searrow & \downarrow {\scriptstyle \mu} \\ & & G \end{array} \qquad \text{(Left inverse)}$$

$$\begin{array}{ccc} G & \xrightarrow{(\mathrm{id},i)} & G \times G \\ & {\scriptstyle e_G} \searrow & \downarrow {\scriptstyle \mu} \\ & & G \end{array} \qquad \text{(Right inverse)}$$

where $e_G \colon G \to G$ is the composite $X \to 1 \xrightarrow{e} X$.

Example 1.2. A group object in the category **Set** is just a group.

Notation 1.3. The category of group objects in $\mathcal{C}$ is denoted $\mathbf{Gp}(\mathcal{C})$. Morphisms of group objects are morphisms in $\mathcal{C}$ that commute with the structure maps.

There is the forgetful functor $U \colon \mathbf{Gp}(\mathcal{C}) \to \mathcal{C}$ which remembers the underlying object but forgets the structure maps.

Proposition 1.4. *Let $\mathcal{C}$ be a locally small category with finite products, including a terminal object. Let G be a group object in $\mathcal{C}$. Then for any object X of $\mathcal{C}$, the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, G)$ is naturally a group.*

In other words, the structure maps of G induce a group structure on $\mathrm{Hom}_{\mathcal{C}}(X, G)$, and this assignment

$$\mathrm{Hom}_{\mathcal{C}}(-, G) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Gp}$$

is a functor.

Proof. Homework 1 Problem 2. □

Remark 1.5. Several authors *define* a group object in an arbitrary locally small category $\mathcal{C}$ as an object G of $\mathcal{C}$ together with a lift of the functor $\mathrm{Hom}_{\mathcal{C}}(-, G) \colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ to groups, as illustrated in the diagram

$$\begin{array}{ccc} & & \mathbf{Gp} \\ & \nearrow & \downarrow {\scriptstyle U} \\ \mathcal{C}^{\mathrm{op}} & \xrightarrow[\mathrm{Hom}_{\mathcal{C}}(-,G)]{} & \mathbf{Set}. \end{array}$$

This definition becomes equivalent to Definition 1.1 when $\mathcal{C}$ has finite limits.

2 Cogroup objects

Definition 2.1. Let $\mathcal{C}$ be a category with finite coproducts, including an initial object $\emptyset$. A **cogroup object** in $\mathcal{C}$ is a group object in the opposite category $\mathcal{C}^{\mathrm{op}}$.

More explicitly, it consists of an object C of $\mathcal{C}$ equipped with a comultiplication $C \to C \amalg C$, counit $C \to \emptyset$, and coinverse $C \to C$, satisfying coassociativity, etc.

Example 2.2. The only cogroup object in **Set** (or in **Top**) is the empty set $\emptyset$, because it is the only object C admitting a map $C \to \emptyset$ to the empty set, which is the initial object.

Definition 2.3. A **homotopy group object** in $\mathcal{C} = \mathbf{Top}$ or $\mathbf{Top}_*$ (or any category with a good notion of homotopy between maps) is defined like a group object, except that the diagrams are only required to commute up to homotopy.

In particular, a homotopy group object in $\mathcal{C}$ becomes a group object in the homotopy category $\mathrm{Ho}(\mathcal{C})$.

A **homotopy cogroup object** in $\mathcal{C}$ is defined similarly.