Math 527 - Homotopy Theory
Additional notes

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The category **Top** is not Cartesian closed. In these notes, we explain how to remedy that problem.

1 Compactly generated spaces

This section and the next are essentially taken from [3, §1,2].

1.1 Basic definitions and properties

Definition 1.1. Let X be a topological space. A subset $A \subseteq X$ is called k**-closed** in X if for any compact Hausdorff space K and continuous map $u \colon K \to X$, the preimage $u^{-1}(A) \subseteq K$ is closed in K.

The collection of k-closed subsets of X forms a topology, which contains the original topology of X (i.e. closed subsets are always k-closed).

Notation 1.2. Let kX denote the space whose underlying set is that of X, but equipped with the topology of k-closed subsets of X. Because the k-topology contains the original topology on X, the identity function $\mathrm{id} \colon kX \to X$ is continuous.

Definition 1.3. A space X is **compactly generated** (CG), sometimes called a **k-space**, if $kX \to X$ is a homeomorphism. In other words, every k-closed subset of X is closed in X.

Example 1.4. Every locally compact space is CG.

Example 1.5. Every first-countable space is CG. More generally, every sequential space is CG.

Example 1.6. Every CW-complex is CG.

Notation 1.7. Let **CG** denote the full subcategory of **Top** consisting of compactly generated spaces.

Notation 1.8. The construction of kX defines a functor $k \colon \mathbf{Top} \to \mathbf{CG}$, called the k**-ification** functor.

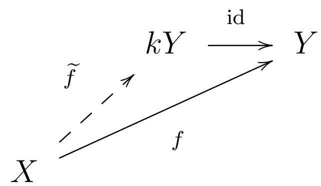
Proposition 1.9. *Let X be a CG space and Y an arbitrary space. Then a function $f \colon X \to Y$ is continuous if and only if for every compact Hausdorff space K and continuous map $u \colon K \to X$, the composite $fu : K \to Y$ is continuous.*

Proposition 1.10. *For any space X, we have $k^2X = kX$, so that kX is always compactly generated.*

Proposition 1.11. *Let X be a CG space and Y an arbitrary space. Then a function $f\colon X \to Y$ is continuous if and only it is continuous when viewed as a function $f\colon X \to kY$.*

Proposition 1.11 can be reformulated in the following more suggestive way, as a universal property.

For any CG space X and continuous map $f\colon X \to Y$, there exists a unique continuous map $\widetilde{f}\colon X \to kY$ satisfying $f = \mathrm{id} \circ \widetilde{f}$, i.e. making the diagram



commute. Note that $\widetilde{f}$ has the same underlying function as f. This exhibits $kY \to Y$ as the "closest approximation" of Y by a CG space.

Corollary 1.12. *The k-ification functor $k\colon$ **Top** $\to$ **CG** is right adjoint to the inclusion $\iota\colon$ **CG** $\to$ **Top**. In other words,* **CG** *is a coreflective subcategory of* **Top**.

The identity function $\iota kX \to X$ is the counit of the adjunction, whereas the unit $W \to k\iota W$ is the identity map for any CG space W.

Proposition 1.13. *1. The category* **CG** *is complete. Limits in* **CG** *are obtained by applying k to the limit in* **Top**.

2. The category **CG** *is cocomplete. Colimits in* **CG** *are computed in* **Top**.

Proof. Let I be a small category and $F\colon I \to \mathbf{CG}$ an I-diagram. Let us write $X_i := F(i)$ and, by abuse of notation, $\lim_i X_i := \lim_I F$.

(1) Viewing the CG spaces X_i as spaces ιX_i, we can compute the limit of the diagram ιF since **Top** is complete (c.f. Homework 4 Problem 2). Applying k yields the CG space $k(\lim_i \iota X_i)$. For any CG space W, we have a natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathbf{CG}}(W, k(\lim_i \iota X_i)) &\cong \mathrm{Hom}_{\mathbf{Top}}(\iota W, \lim_i \iota X_i) \\ &\cong \lim_i \mathrm{Hom}_{\mathbf{Top}}(\iota W, \iota X_i) \\ &= \lim_i \mathrm{Hom}_{\mathbf{CG}}(W, X_i) \end{aligned}$$

where the last equality comes from the fact that **CG** is a full subcategory of **Top**. This proves $k(\lim_i \iota X_i) = \lim_i X_i$.

(2) We can compute the colimit $X = \mathrm{colim}_i\, \iota X_i$ of the diagram ιF since **Top** is cocomplete (c.f. Homework 4 Problem 2 and Remark afterwards). Since X is a quotient of a coproduct of CG spaces ιX_i, X is also CG, by [3, Prop. 2.1, Prop. 2.2]. Moreover it is the desired colimit

in **CG**. For any CG space Y, we have a natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathbf{CG}}(X,Y) &= \mathrm{Hom}_{\mathbf{Top}}(\iota X, \iota Y) \\ &= \mathrm{Hom}_{\mathbf{Top}}(\iota \operatorname*{colim}_i \iota X_i, \iota Y) \\ &= \mathrm{Hom}_{\mathbf{Top}}(\operatorname*{colim}_i \iota X_i, \iota Y) \\ &\cong \lim_i \mathrm{Hom}_{\mathbf{Top}}(\iota X_i, \iota Y) \\ &= \lim_i \mathrm{Hom}_{\mathbf{CG}}(X_i, Y) \end{aligned}$$

which proves $X = \operatorname{colim}_i X_i$. □

In particular, products in CG may not agree with the usual product in **Top**.

Notation 1.14. For CG spaces X and Y, write $X \times_0 Y = \iota X \times \iota Y$ for their usual product in **Top**, and write $X \times Y = k(X \times_0 Y)$ for their product in **CG**.

1.2 Mapping spaces

Definition 1.15. Let X and Y be CG spaces. For any compact Hausdorff space K, continuous map $u\colon K \to X$, and open subset $U \subseteq Y$, consider the set

$$W(u, K, U) := \{f\colon X \to Y \text{ continuous } \mid fu(K) \subseteq U\}.$$

Denote by $C_0(X,Y)$ the set of continuous maps from X to Y, equipped with the topology generated by all such subsets $W(u,K,U)$. This topology is called the **compact-open topology**.

Note that $C_0(X,Y)$ need not be CG. Write $\mathrm{Map}(X,Y) := kC_0(X,Y)$.

Theorem 1.16. *For any CG spaces X, Y, and Z, the natural map*

$$\varphi\colon \mathrm{Map}(X \times Y, X) \to \mathrm{Map}(X, \mathrm{Map}(Y,Z)) \tag{1}$$

is a homeomorphism.

The fact that φ is bijective tells us that **CG** is Cartesian closed, in the unenriched sense. The theorem is even better: **CG** is Cartesian closed, in the enriched sense. Note that **CG** is enriched in itself, given that the composition map

$$\mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z) \xrightarrow{\circ} \mathrm{Map}(X,Z)$$

is continuous.

Remark 1.17. The exponential object $\mathrm{Map}(X,Y)$ is often denoted Y^X. The isomorphism (1), which can be written as

$$Z^{X \times Y} \cong (Z^Y)^X$$

is often called the **exponential law**.

2 Weakly Hausdorff spaces

The category **CG** would be good enough to work with, but we can also impose a separation axiom to our spaces.

Definition 2.1. A topological space X is **weakly Hausdorff** (WH) if for every compact Hausdorff space K and every continuous map $u\colon K \to X$, the image $u(K) \subseteq X$ is closed in X.

Remark 2.2. Hausdorff spaces are weakly Hausdorff, since $u(K)$ is compact and thus closed in X if X is Hausdorff. This justifies the terminology.

Moreover, weakly Hausdorff spaces are T_1, since the single point space $*$ is compact Hausdorff. Thus we have implications

$$\text{Hausdorff} \Rightarrow \text{weakly Hausdorff} \Rightarrow T_1.$$

Example 2.3. Every CW-complex is Hausdorff, hence in particular WH.

Proposition 2.4. *If X is a WH space, then any larger topology on X is still WH. In particular, kX is still WH.*

Proof. Let X' be the set X equipped with a topology containing the original topology, i.e. the identity function $\mathrm{id}\colon X' \to X$ is continuous. For any compact Hausdorff space K and continuous map $u\colon K \to X'$, the composite $\mathrm{id}u\colon K \to X$ is continuous and so its image $\mathrm{id}u(K) \subseteq X$ is closed in X. Thus $u(K) = \mathrm{id}^{-1}\mathrm{id}u(K)$ is closed in X'. □

Proposition 2.5. *Any subspace of a WH space is WH.*

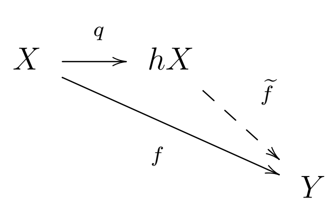
Proof. Let X be a WH space and $i\colon A \hookrightarrow X$ the inclusion of a subspace. For any compact Hausdorff space K and continuous map $u\colon K \to A$, the composite $iu\colon K \to X$ is continuous and so its image $iu(K) \subseteq X$ is closed in X, and thus in A as well. □

Notation 2.6. Let **CGWH** denote the full subcategory of **CG** consisting of compactly generated weakly Hausdorff spaces.

Definition 2.7. For any CG space X, let hX be the quotient of X by the smallest closed equivalence relation on X (see [3, Prop. 2.22]). Then hX is still CG since it is a quotient of a CG space [3, Prop. 2.1], and it is WH since we quotiented out a closed equivalence relation on X [3, Cor. 2.21].

This defines a functor $h\colon \mathbf{CG} \to \mathbf{CGWH}$ called **weak Hausdorffification**

By construction, the quotient map $q\colon X \twoheadrightarrow hX$ satisfies the following universal property. For any CGWH space Y and continuous map $f\colon X \to Y$, there exists a unique continuous map $\widetilde{f}\colon hX \to Y$ satisfying $f = \widetilde{f}q$, i.e. making the diagram



commute. This exhibits $X \to hX$ as the "closest approximation" of X by a CGWH space.

Corollary 2.8. *The functor* $h\colon \mathbf{CG} \to \mathbf{CGWH}$ *is left adjoint to the inclusion functor* $\iota\colon \mathbf{CG} \to \mathbf{CGWH}$. *In other words,* $\mathbf{CGWH}$ *is a reflective subcategory of* $\mathbf{CG}$.

The quotient map $q\colon X \twoheadrightarrow hX$ *is the unit of the adjunction, whereas the counit* $h\iota W \to W$ *is the identity map for any CGWH space* W.

Proposition 2.9.

1. *The category* $\mathbf{CGWH}$ *is complete. Limits in* $\mathbf{CGWH}$ *are computed in* $\mathbf{CG}$.
2. *The category* $\mathbf{CGWH}$ *is cocomplete. Colimits in* $\mathbf{CGWH}$ *are obtained by applying* h *to the colimit in* $\mathbf{CG}$.

Proof. Let I be a small category and $F\colon I \to \mathbf{CGWH}$ an I-diagram.

(1) The limit $X = \lim_i \iota X_i$ computed in $\mathbf{CG}$, which exists since $\mathbf{CG}$ is complete (by 1.13), is still WH. Indeed, an arbitrary product in $\mathbf{CG}$ of CGWH spaces is still WH [3, Cor. 2.16], and so is an equalizer in $\mathbf{CG}$ of two maps (by 2.5 and 2.4). Therefore X is also the limit in $\mathbf{CGWH}$, by the same argument as 1.13 (2).

(2) We have $h(\operatorname{colim}_i \iota X_i) = \operatorname{colim} X_i$ in $\mathbf{CGWH}$ by the same argument as 1.13 (1). $\square$

To summarize the situation, there are two adjoint pairs as follows:

$$\begin{array}{ccc} \mathbf{CG} & \underset{\iota}{\overset{h}{\rightleftarrows}} & \mathbf{CGWH} \\ {\scriptstyle \iota}\downarrow\uparrow{\scriptstyle k} & & \\ \mathbf{Top} & & \end{array}$$

Proposition 2.10. *If* X *is a CG space and* Y *is a CGWH space, then* $\operatorname{Map}(X, Y)$ *is CGWH.*

Consequently, the category $\mathbf{CGWH}$ is enriched in itself. Note that it is also Cartesian closed (in the enriched sense). Indeed, for any X, Y, and Z in $\mathbf{CGWH}$, the natural map

$$\varphi\colon \operatorname{Map}(X \times Y, X) \to \operatorname{Map}(X, \operatorname{Map}(Y, Z))$$

is a homeomorphism.

3 A convenient category of spaces

In this section, we explain in what sense it is preferable to work with the category **CGWH** instead of **Top**. We follow the treatment in [1], itself inspired by [2].

Definition 3.1. A **convenient category of topological spaces** is a full replete (meaning closed under isomorphisms of objects) subcategory $\mathcal{C}$ of **Top** satisfying the following conditions.

1. All CW-complexes are objects of $\mathcal{C}$.
2. $\mathcal{C}$ is complete and cocomplete.
3. $\mathcal{C}$ is Cartesian closed.

Note that **CG** and **CGWH** are replete full subcategories of **Top**, since both conditions of being CG or WH are invariant under homeomorphism. Let us summarize the discussion as follows.

Proposition 3.2. *The categories* **CG** *and* **CGWH** *are convenient.*

In fact, there are other desirable properties for a convenient category of spaces. For instance, one would like that closed subspaces of objects in $\mathcal{C}$ also be in $\mathcal{C}$. Both **CG** and **CGWH** satisfy this additional condition.

Proposition 3.3. *Consider inclusions of spaces $A \subseteq B \subseteq X$. If A is k-closed in B and B is k-closed in X, then A is k-closed in X.*

Proof. Let K be a compact Hausdorff space and $u\colon K \to X$ a continuous map. Then the preimage $u^{-1}(B)$ is closed in K, hence compact (and also Hausdorff). Consider the restriction $u|_{u^{-1}(B)}\colon u^{-1}(B) \to B$. Since A is k-closed in B, the preimage $u|_{u^{-1}(B)}^{-1}(A) = u^{-1}(A)$ is closed in $u^{-1}(B)$ and thus in K as well. □

Corollary 3.4. *A closed subspace of a CG space is also CG.*

Proof. Let X be a CG space and $B \subseteq X$ a closed subspace. Let $A \subseteq B$ be a k-closed subset of B. Then A is k-closed in X (since B is closed in X), hence closed in X (since X is CG). Therefore A is closed in B. □

Corollary 3.5. *A closed subspace of a CGWH space is also CGWH.*

4 Some applications

Here is a toy example illustrating the use of the exponential law.

Proposition 4.1. *For any "spaces" X and Y, the natural map* $[X, Y] \xrightarrow{\cong} \pi_0 \operatorname{Map}(X, Y)$ *is a bijection.*

Proof. Since the map

$$\operatorname{Map}(X \times I, Y) \to \operatorname{Map}(I, \operatorname{Map}(X, Y))$$

is a bijection, two maps $f, g\colon X \to Y$ are homotopic if and only if they are connected by a (continuous) path in $\operatorname{Map}(X, Y)$. □

Here is another benefit of working in the category **CG**.

Proposition 4.2. *If X and Y are CW-complexes, then $X \times Y$ inherits a CW-structure, where a p-cell of X and a q-cell of Y produce a $(p + q)$-cell of $X \times Y$.*

Here we do mean the product $X \times Y$ is **CG**, not in **Top**. A priori, the product $X \times_0 Y$ in **Top** could fail to be a CW-complex. See Hatcher (A.6) for details.

References

[1] Ronald Brown, *nLab: Convenient category of topological spaces* (2012), available at `http://ncatlab.org/nlab/show/convenient+category+of+topological+spaces`.

[2] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133 –152.

[3] Neil Strickland, *The category of CGWH spaces* (2009), available at `http://www.neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf`.