Math 527 - Homotopy Theory
Eilenberg-MacLane spaces and cohomology

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1 Background material

Proposition 1.1. *Let $n \geq 1$ and let G be an abelian group. For any abelian group M, the cohomology of $K(G,n)$ with coefficients in M satisfies*

$$H^n\left(K(G,n);M\right) \cong \mathrm{Hom}_{\mathbb{Z}}(G,M).$$

Proof. By the Hurewicz theorem, we have

$$H_n\left(K(G,n);\mathbb{Z}\right) \cong \pi_n\left(K(G,n)\right) \cong G.$$

By the universal coefficient theorem, we have

$$\begin{aligned} H^n\left(K(G,n);M\right) &\cong \mathrm{Hom}_{\mathbb{Z}}\left(H_n(K(G,n);\mathbb{Z}),M\right) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(G,M). \end{aligned}$$

□

Definition 1.2. Let $n \geq 1$ and let G be an abelian group. The **fundamental class** of $K(G,n)$ is the cohomology class

$$\iota_n \in H^n\left(K(G,n);G\right)$$

corresponding to id_G via the isomorphism $H^n\left(K(G,n);G\right) \cong \mathrm{Hom}_{\mathbb{Z}}(G,G)$.

More explicitly, let $\psi\colon \pi_n K(G,n) \xrightarrow{\cong} G$ be some chosen identification, and let

$$h\colon \pi_n\left(K(G,n)\right) \xrightarrow{\cong} H_n\left(K(G,n);\mathbb{Z}\right)$$

denote the Hurewicz morphism, defined by $h(\alpha) = \alpha_*(u_n)$, where $u_n \in H_n(S^n)$ is a suitably chosen generator. Then ι_n is defined by the equation

$$\langle \iota_n, h(\alpha) \rangle = \psi(\alpha)$$

for all $\alpha \in \pi_n K(G, n)$. Here the brackets denote the evaluation pairing

$$\langle -, - \rangle \colon H^n(W; G) \otimes_{\mathbb{Z}} H_n(W; \mathbb{Z}) \to G$$

between cohomology and homology.

For any map $\alpha \colon X \to K(G, n)$, consider the induced "pullback" map on cohomology

$$\alpha^* \colon H^n (K(G, n); G) \to H^n(X; G)$$

and take the pullback of the fundamental class $\alpha^*(\iota_n) \in H^n(X; G)$. Since cohomology is homotopy invariant, this defines a function

$$\begin{aligned} \theta_X \colon [X, K(G, n)]_* &\to H^n(X; G) \\ \alpha &\mapsto \alpha^*(\iota_n) \end{aligned}$$

which is natural in X.

Proposition 1.3. *The map θ_X is a group homomorphism.*

Proof. Hatcher § 4.3 Exercise 7. □

Some of the arguments will require working with $n = 0$, which we treat separately.

For any space X and abelian group M, the zeroth cohomology is

$$\begin{aligned} H^0 (X; M) &\cong \mathrm{Hom}_{\mathbb{Z}} (H_0(X; \mathbb{Z}), M) \\ &\cong \mathrm{Hom}_{\mathbb{Z}} (\mathbb{Z}\langle \pi_0(X) \rangle, M) \\ &\cong \mathrm{Hom}_{\mathbf{Set}} (\pi_0(X), M) \\ &\cong \prod_{\pi_0(X)} M. \end{aligned}$$

Here $\mathbb{Z}\langle \pi_0(X) \rangle$ denotes the free abelian group on the set $\pi_0(X)$.

Definition 1.4. Let G be an abelian group. The **fundamental class** of $K(G, 0)$ is the cohomology class

$$\iota_0 \in H^0 (K(G, n); G)$$

corresponding to id_G via the isomorphism $H^0 (K(G, n); G) \cong \mathrm{Hom}_{\mathbf{Set}}(G, G)$.

We also denote by $\iota_0 \in \widetilde{H}^0 (K(G, n); G)$ its image via the canonical quotient map

$$H^0 (K(G, n); G) \twoheadrightarrow \widetilde{H}^0 (K(G, n); G) .$$

Proposition 1.5. *Let $n = 0$ and assume X is the coproduct of its path components (which holds in particular if X is locally path-connected, in particular if X a CW complex). Then the maps*

$$\theta \colon [X, K(G, 0)] \to H^0(X; G)$$

$$\theta \colon [X, K(G, 0)]_* \to \widetilde{H}^0(X; G)$$

defined by pulling back the fundamental class ι_0 are isomorphisms.

Proof. Since X is the coproduct of its path-components and $K(G,0)$ is homotopically discrete, we have

$$[X, K(G,0)] \cong \mathrm{Hom}_{\mathbf{Set}}\left(\pi_0(X), G\right) \cong \prod_{\pi_0(X)} G$$

$$[X, K(G,0)]_* \cong \mathrm{Hom}_{\mathbf{Set}_*}\left(\pi_0(X), G\right) \cong \prod_{\pi_0(X)\backslash\{C_0\}} G$$

where $C_0 \in \pi_0(X)$ denotes the basepoint component.

The right-hand sides $H^0(X;G)$ and $\widetilde{H}^0(X;G)$ are also naturally isomorphic to those respective products. One readily checks that θ induces an isomorphism (of abelian groups). $\square$

2 Main statements

Lemma 2.1. *Let $n \geq 1$ and assume $X = \bigvee_{j\in J} S^n$ is a wedge of n-spheres. Then*

$$\theta \colon [\bigvee_j S^n, K(G,n)]_* \xrightarrow{\cong} \widetilde{H}^n(\bigvee_j S^n; G)$$

is an isomorphism.

Proof. **Step 1. Single sphere.** Let $\psi\colon \pi_n K(G,n) \cong G$ be an identification as in the definition of θ. Then the composite

$$\begin{tikzcd}
{[S^n, K(G,n)]_*} \arrow[r, "\theta_{S^n}"] & \widetilde{H}^n(S^n;G) \arrow[d, "\cong"'] & \alpha^*(\iota_n) \arrow[d, mapsto] \\
& \mathrm{Hom}_{\mathbb{Z}}\left(\widetilde{H}_n(S^n), G\right) \arrow[d, "\cong"', "\mathrm{ev}_{u_n}"] & \langle \alpha^*(\iota_n), - \rangle = \langle \iota_n, \alpha_*(-) \rangle \arrow[d, mapsto] \\
& G & \langle \iota_n, \alpha_*(u_n) \rangle = \langle \iota_n, h(\alpha) \rangle = \psi(\alpha)
\end{tikzcd}$$

is the isomorphism ψ. Therefore θ_{S^n} is an isomorphism.

Step 2. Arbitrary wedge. Both functors $[-, K(G,n)]_*$ and $\widetilde{H}^n(-;G)$ take wedges to products, so that the case of a single sphere proves the statement, as illustrated in the commutative diagram:

$$\begin{tikzcd}
{[\bigvee_j S^n, K(G,n)]_*} \arrow[r, "\theta_{\bigvee_j S^n}", "\therefore\cong"'] \arrow[d, "\cong"'] & \widetilde{H}^n(\bigvee_j S^n; G) \arrow[d, "\cong"] \\
\prod_j [S^n, K(G,n)]_* \arrow[r, "\cong", "\prod_j \theta_{S^n}"'] & \prod_j \widetilde{H}^n(S^n;G)
\end{tikzcd}$$

$\square$

Proposition 2.2. *Let $n \geq 1$. Then the diagram*

$$\begin{tikzcd}
& {[\Sigma X, K(G,n)]_*} \arrow[r, "\theta^n_{\Sigma X}"] & \widetilde{H}^n(\Sigma X; G) \\
{[X, K(G,n-1)]_* \cong [X, \Omega K(G,n)]_*} \arrow[ur, "\cong"] \arrow[rr, "\theta^{n-1}_X"'] & & \widetilde{H}^{n-1}(X;G) \arrow[u, "\Sigma"']
\end{tikzcd}$$

commutes up to sign.

In particular, if X has the homotopy type of a CW complex (so that the suspension isomorphism on homology holds), then the bottom map θ_X^{n-1} is an isomorphism if and only if the top map $\theta_{\Sigma X}^n$ is.

Theorem 2.3. *Let $n \geq 0$ and G be an abelian group. If X is a CW complex, then the natural map*

$$\theta_X \colon [X, K(G,n)]_* \xrightarrow{\cong} \widetilde{H}^n(X;G)$$

is an isomorphism.

Proof. The case $n = 0$ has already been proved in 1.5, so we may assume $n \geq 1$. In that case, we may assume X is path-connected WLOG.

Step 1. Induction on skeleta. We will prove the statement by induction on the dimension k of the CW complex X_k. For dimension $k = 0$, $\theta_{X_0} \colon 0 \to 0$ is trivially an isomorphism.

Now assume that the statement holds for all CW complexes of dimension less than k. Consider the cofiber sequence

$$\bigvee S^{k-1} \longrightarrow X_{k-1} \longrightarrow X_k \longrightarrow \bigvee S^k \longrightarrow \Sigma X_{k-1}.$$

Applying the natural map θ and writing $K := K(G,n)$, we obtain a map of exact sequences

$$\begin{tikzcd}
{[\Sigma X_{k-1}, K]_*} \arrow[r] \arrow[d] & {[\bigvee S^k, K]_*} \arrow[r] \arrow[d, "\cong"'] & {[X_k, K]_*} \arrow[r] \arrow[d] & {[X_{k-1}, K]_*} \arrow[r] \arrow[d, "\cong"'] & {[\bigvee S^{k-1}, K]_*} \arrow[d, "\cong"'] \\
\widetilde{H}^n(\Sigma X_{k-1}; G) \arrow[r] & \widetilde{H}^n(\bigvee S^k; G) \arrow[r] & \widetilde{H}^n(X_k; G) \arrow[r] & \widetilde{H}^n(X_{k-1}; G) \arrow[r] & \widetilde{H}^n(\bigvee S^{k-1}; G)
\end{tikzcd}$$

where θ is an isomorphism for wedges of spheres by 2.1 and for X_{k-1} by induction hypothesis. By the four-lemma for epimorphisms, θ_{X_k} is an epimorphism. Since this argument works simultaneously for all CW complexes of dimension at most k, it also applies to ΣX_{k-1} so that $\theta_{\Sigma X_{k-1}}$ is an epimorphism. By the four-lemma for monomorphisms, θ_{X_k} is a monomorphism, and thus an isomorphism.

Step 2. The induction stops. Consider the skeletal inclusion $X_{n+1} \hookrightarrow X$ and the induced restriction maps

$$\begin{tikzcd}
{[X, K(G,n)]_*} \arrow[r, "\theta_X"] \arrow[d] & \widetilde{H}^n(X; G) \arrow[d] \\
{[X_{n+1}, K(G,n)]_*} \arrow[r, "\cong", "\theta_{X_{n+1}}"'] & \widetilde{H}^n(X_{n+1}; G)
\end{tikzcd}$$

where the bottom map is an isomorphism by Step 1. We want to show that the top map θ_X is an isomorphism. It suffices to show that both downward restriction maps are isomorphisms.

The restriction map $\widetilde{H}^n(X;G) \xrightarrow{\cong} \widetilde{H}^n(X_{n+1};G)$ is an isomorphism, by cellular cohomology.

The restriction map $[X, K(G,n)]_* \to [X_{n+1}, K(G,n)]_*$ is surjective, i.e. any map $f\colon X_{n+1} \to K(G,n)$ can be extended to X. Indeed, for any $(n+2)$-cell with attaching map $\varphi\colon S^{n+1} \to X$, the composite $f \circ \varphi\colon S^{n+1} \to K(G,n)$ is null, by the condition $\pi_{n+1}K(G,n) = 0$. Thus f can be extended to X_{n+2} and likewise for all higher skeleta.

The restriction map $[X, K(G,n)]_* \to [X_{n+1}, K(G,n)]_*$ is injective. Assume two maps $f, g\colon X \to K(G,n)$ have homotopic restrictions $f|_{X_{n+1}} \simeq g|_{X_{n+1}}$ via a (pointed) homotopy

$$F\colon X_{n+1} \wedge I_+ \to K(G,n).$$

Since $X_{n+1} \wedge I_+ \cup X \wedge \partial I_+ \subseteq X \wedge I_+$ is a subcomplex containing all cells of dimension up to $n+1$, the remaining cells of $X \wedge I_+$ have dimension at least $n+2$. Hence, the same argument as above allows to extend the homotopy F to a homotopy

$$\overline{F}\colon X \wedge I_+ \to K(G,n).$$

between f and g. $\square$

More details can be found in tom Dieck § 17.5, particularly Theorem 17.5.1.

Remark 2.4. One can weaken the assumption to X being well-pointed and having the homotopy type of a CW complex. Indeed, the functor $\widetilde{H}^n(-;G)$ is homotopy invariant, while the functor $[-, K(G,n)]_*$ is invariant under pointed homotopy equivalence.

A digression

To show that the restriction $[X, K(G,n)]_* \to [X_{n+1}, K(G,n)]_*$ is a bijection, one might be tempted by the following argument. Consider the cofiber sequence

$$\bigvee S^{n+1} \to X_{n+1} \to X_{n+2} \to \bigvee S^{n+2}$$

and apply the functor $[-, K]_*$, yielding the exact sequence

$$[\bigvee S^{n+2}, K]_* \to [X_{n+2}, K]_* \to [X_{n+1}, K]_* \to [\bigvee S^{n+1}, K]_*.$$

The outer terms are both trivial:

$$[\bigvee S^{n+2}, K]_* \cong \prod \pi_{n+2} K = 0$$
$$[\bigvee S^{n+1}, K]_* \cong \prod \pi_{n+1} K = 0$$

and thus the exact sequence proves the bijectivity of $[X_{n+2}, K]_* \xrightarrow{\simeq} [X_{n+1}, K]_*$. The same argument applies to all higher skeleta, proving that the map

$$\lim_k [X_k, K]_* \xrightarrow{\simeq} [X_{n+1}, K]_*$$

is a bijection.

However, the natural map

$$[X, Y]_* \to \lim_k [X_k, Y]_*$$

is **not** in general a bijection. Rather, it sits in an exact sequence

$$0 \to \lim_k{}^1 [\Sigma X_k, Y]_* \to [X, Y]_* \to \lim_k [X_k, Y]_* \to 0$$

so that the argument above only shows surjectivity, not injectivity. Maps $X \to Y$ in the kernel are those that become null when restricted to every skeleton X_k, called (skeletally) **phantom maps**. See May-Ponto Proposition 2.1.9 and Corollary 2.1.14, as well as § 2.4 which contains a proof that there are uncountably many phantom maps $\mathbb{C}P^\infty \to S^3$.

The issue here is that the equation $X \cong \operatorname{colim}_k X_k$ holds in **Top** and $\mathbf{Top}_*$, but **not** in the homotopy category Ho(**Top**) or Ho($\mathbf{Top}_*$). Since each skeletal inclusion is a cofibration, X is in fact the homotopy colimit of its skeleta $X \simeq \operatorname{hocolim}_k X_k$, which is very different from being a colimit in the homotopy category.

3 Alternate proofs

In this section, we outline other proofs of the main theorem 2.3.

3.1 Eilenberg-Steenrod axioms

Proof. One can show that the functors $h^n := [-, K(G, n)]_*$ satisfy the Eilenberg-Steenrod axioms for a reduced cohomology theory, including the dimension axiom $h^n(\text{point}) = 0$ for all n.

By uniqueness, we conclude that $[-, K(G, n)]_*$ is naturally isomorphic to ordinary reduced cohomology $\widetilde{H}^n\left(-; h^0(S^0)\right)$ with coefficients

$$\begin{aligned} h^0(S^0) &= [S^0, K(G, 0)]_* \\ &= \pi_0 K(G, 0) \\ &\cong G. \end{aligned}$$

□

3.2 Brown representability

Proof. Reduced cohomology $\widetilde{H}^n(-; G)$, as a functor $\mathbf{CW}_{*,0}^{\mathrm{op}} \to \mathbf{Set}$ on the category of pointed connected CW complexes, satisfies the assumptions of Brown representability. Therefore it is represented by an object K_n, i.e. satisfying a natural isomorphism

$$\widetilde{H}^n(X; G) \cong [X, K_n]_*.$$

Evaluating the functor on spheres S^k for $k \geq 1$, we obtain

$$\begin{aligned} \pi_k(K_n) &= [S^k, K_n]_* \\ &\cong \widetilde{H}^n(S^k; G) \\ &\cong \begin{cases} G & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases} \end{aligned}$$

so that K_n is an Eilenberg-MacLane space $K_n \simeq K(G, n)$. Note that the isomorphism

$$\widetilde{H}^0(X; G) \cong [X, K(G, 0)]_*$$

has already been established directly.

To conclude, note that by Yoneda, the natural isomorphism of functors has the form

$$\begin{aligned} [X, K_n]_* &\cong \widetilde{H}^n(X; G) \\ \alpha &\mapsto \alpha^*(\iota_n) \end{aligned}$$

for the class $\iota_n \in \widetilde{H}^n(K_n)$ corresponding to $\mathrm{id} \in [K_n, K_n]_*$. □

3.3 Cellular cohomology and obstruction theory

Proof. Let $n \geq 1$. WLOG the space $K(G,n)$ is a CW complex with a single 0-cell and no cells in dimensions $0 < i < n$.

Any map $f: X \to K(G,n)$ is homotopic to a cellular map, and hence constant on the skeleton X_{n-1}. If f is constant on X_{n-1}, consider its restriction

$$f|_{X_n}: X_n/X_{n-1} \cong \bigvee_{n\text{-cells of} X} S^n \to K(G,n)$$

which defines an element of $\pi_n K(G,n) \cong G$ for each n-cell of X, i.e. a cellular n-cochain $\kappa(f) \in C^n_{CW}(X;G)$ with coefficients in G.

Two maps $f, f': X \to K(G,n)$ are homotopic if and only if they are homotopic rel X_{n-2}.

Two maps $f, f': X_n \to K(G,n)$ that send X_{n-1} to the basepoint are homotopic rel X_{n-2} if and only if the corresponding cochains $\kappa(f)$ and $\kappa(f')$ differ by a coboundary. The "only if" direction guarantees that the function

$$\kappa: [X_n, K(G,n)]_* \to C^n_{CW}(X;G)/B^n_{CW}(X;G)$$

is well defined. The "if" direction guarantees that it is injective.

A map $f: X_n \to K(G,n)$ extends to X_{n+1} if and only if the corresponding cochain $\kappa(f)$ is a cocycle, i.e. satisfies $\delta\kappa(f) = 0 \in C^{n+1}_{CW}(X;G)$. The "only if" direction guarantees that the top map in the diagram

$$\begin{tikzcd}
{[X_{n+1}, K(G,n)]_*} \arrow[r, two heads, "\kappa"] \arrow[d, "\mathrm{res}"'] & Z^n_{CW}(X;G)/B^n_{CW}(X;G) \arrow[d, hook] \\
{[X_n, K(G,n)]_*} \arrow[r, hook, "\kappa"'] & C^n_{CW}(X;G)/B^n_{CW}(X;G)
\end{tikzcd}$$

exists. The "if" direction guarantees that it is surjective.

An extension of $f: X_n \to K(G,n)$ to X_{n+1}, when it exists, is unique up to homotopy, as a consequence of $\pi_{n+1}K(G,n) = 0$. Therefore the restriction

$$\mathrm{res}: [X_{n+1}, K(G,n)]_* \hookrightarrow [X_n, K(G,n)]_*$$

is injective, and so is the top map

$$\kappa: [X_{n+1}, K(G,n)]_* \xrightarrow{\simeq} Z^n_{CW}(X;G)/B^n_{CW}(X;G) = H^n_{CW}(X;G).$$

Any map $f: X_{n+1} \to K(G,n)$ extends to X, and the extension is unique up to homotopy. This is again a consequence of $\pi_i K(G,n) = 0$ for all $i > n$.

Combining all this, we obtain the isomorphism

$$[X, K(G,n)]_* \xrightarrow[\simeq]{\mathrm{res}} [X_{n+1}, K(G,n)]_* \xrightarrow[\simeq]{\kappa} H^n_{CW}(X;G)$$

to cellular cohomology of X. $\square$

More details can be found here:

http://mathoverflow.net/questions/5518/dirty-proof-that-eilenberg-maclane-spaces-represent-cohomology

as well as in Mosher-Tangora Chapter 1.

4 A fun application

Proposition 4.1. *The degree* 2 *map* $S^1 \xrightarrow{2} S^1$ *admits no cokernel in the homotopy category of pointed spaces* $\mathrm{Ho}(\mathbf{Top}_*)$.

Proof. Assume to the contrary that there exists a cokernel

$$S^1 \xrightarrow{2} S^1 \to X.$$

For any abelian group G and $n \geq 0$, consider the Eilenberg-MacLane $K(G,n)$ and apply the functor $[-, K(G,n)]_*$ to the cokernel sequence above. This yields a kernel sequence

$$[X, K(G,n)]_* \to [S^1, K(G,n)]_* \xrightarrow{2^*} [S^1, K(G,n)]_*$$

from which we deduce

$$\begin{aligned}
\widetilde{H}^n(X;G) &\cong \{\alpha \in \widetilde{H}^n(S^1;G) \mid \alpha \circ 2 = 0\} \\
&= \{\alpha \in \widetilde{H}^n(S^1;G) \mid 2\alpha = 0\} \\
&= \begin{cases} {}_2G & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}
\end{aligned}$$

Here ${}_2G := \{g \in G \mid 2g = 0\}$ denotes the 2-torsion in G. In particular, we obtain the cohomology groups

$$\begin{aligned}
&\widetilde{H}^1(X;\mathbb{Z}) = 0 \\
&\widetilde{H}^1(X;\mathbb{Z}/2) = \mathbb{Z}/2.
\end{aligned}$$

Now consider the short exact sequence of coefficients

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

and the induced long exact sequence on cohomology

$$\ldots \to H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{Z}/2) \xrightarrow{\delta} H^2(X;\mathbb{Z}) \to \ldots \tag{1}$$

whose terms can be rewritten as

$$\ldots \to 0 \to 0 \to \mathbb{Z}/2 \xrightarrow{\delta} 0 \to \ldots.$$

This sequence cannot be exact, providing a contradiction. □

There is a slight problem in this proof. Technically, the identification $[X, K(G,n)]_* \cong \widetilde{H}^n(X;G)$ requires that X have the homotopy type of a CW complex. In the argument above, we really mean $[X, K(G,n)]_*$ the whole time. The functors $[-, K(G,n)]_*$ also have the long exact sequence (1), as we now show.

Proposition 4.2. *Let*

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of abelian groups. Let $K(-,n)$ be a functorial construction of Eilenberg-MacLane spaces as CW complexes. Then the resulting sequence

$$K(A,n) \to K(B,n) \to K(C,n)$$

is a fiber sequence.

Proof. WLOG the construction $K(-,n)$ sends the zero morphism to the constant map. In particular, the composite $A \to C$ is zero, so that the composite $K(A,n) \to K(C,n)$ is the constant map. This yields a canonical map to the homotopy fiber F of $K(B,n) \to K(C,n)$ as illustrated in the commutative diagram

$$\begin{tikzcd} K(A,n) \arrow[r] \arrow[d] & K(B,n) \arrow[r] & K(C,n). \\ F \arrow[ur] & & \end{tikzcd}$$

The long exact sequence on homotopy of the fibration $F \to K(B,n) \to K(C,n)$ shows that $K(A,n) \xrightarrow{\sim} F$ is a weak homotopy equivalence. However, it is a fact that the homotopy fiber of a map between spaces having the homotopy type of CW complexes also has the homotopy type of a CW complex [2, Proposition 12] [1, Theorem 3]. In particular, F has the homotopy type of a CW complex. By the Whitehead theorem, $K(A,n) \xrightarrow{\simeq} F$ is a homotopy equivalence. □

The iterated fiber sequence has the form

$$\ldots \longrightarrow \Omega K(B,n) \longrightarrow \Omega K(C,n) \longrightarrow K(A,n) \longrightarrow K(B,n) \longrightarrow K(C,n)$$

which can be rewritten as

$$* \longrightarrow K(A,0) \longrightarrow K(B,0) \longrightarrow K(C,0) \longrightarrow K(A,1) \longrightarrow K(B,1) \longrightarrow K(C,1) \longrightarrow$$

$$\ldots \longrightarrow K(C,n-1) \longrightarrow K(A,n) \longrightarrow K(B,n) \longrightarrow K(C,n).$$

Applying the functor $[X,-]_*$ to this fiber sequence yields the desired long exact sequence, up to signs of the maps.

5 Pointed versus unpointed

Question 5.1. When is cohomology also given by *unpointed* maps to an Eilenberg-MacLane space?

The case $n = 0$ stated in 1.5 is very misleading. One may think that in general, pointed maps correspond to reduced cohomology while unpointed maps correspond to unreduced cohomology. As we will see shortly, that is very false for $n = 1$ and irrelevant for $n > 1$.

Let us study the natural map $\eta\colon [X, Y]_* \to [X, Y]$ which forgets pointedness. Recall the following basic fact about well-pointed spaces.

Proposition 5.2. *Let (X, x_0) be a well-pointed space and (Y, y_0) any pointed space. Then a map $f\colon X \to Y$ is (freely) homotopic to a pointed map if and only if $f(x_0)$ is in the path component of y_0.*

Though this fact is readily proved directly, we will recover it – and more – using a different perspective.

If (X, x_0) is well-pointed, i.e. the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration, then applying $\mathrm{Map}(-, Y)$ yields a fibration

$$\mathrm{Map}(X, Y) \xrightarrow{\mathrm{ev}_{x_0}} Y$$

which evaluates at the basepoint $x_0 \in X$. Note that ev_{x_0} is surjective. The strict fiber of ev_{x_0} is

$$\mathrm{ev}_{x_0}^{-1}(y_0) = \mathrm{Map}_*(X, Y)$$

the subspace consisting of pointed maps. We obtain a fiber sequence

$$\mathrm{Map}_*(X, Y) \xrightarrow{\iota} \mathrm{Map}(X, Y) \xrightarrow{\mathrm{ev}_{x_0}} Y$$

and therefore a long exact sequence of homotopy groups

$$\ldots \to \pi_1 \mathrm{Map}(X, Y) \to \pi_1 Y \xrightarrow{\partial} \pi_0 \mathrm{Map}_*(X, Y) \to \pi_0 \mathrm{Map}(X, Y) \to \pi_0(Y) \to 0$$

whose last terms can be rewritten as

$$\ldots \to \pi_1 Y \xrightarrow{\partial} [X, Y]_* \xrightarrow{\eta} [X, Y] \to \pi_0(Y) \to 0. \tag{2}$$

Proposition 5.2 was stating the exactness of this sequence at $[X, Y]$.

Exactness at $[X, Y]_*$ says that two pointed maps $f, g\colon X \to Y$ are freely homotopic if and only if they are in the same orbit under the action of $\pi_1(Y)$. See Hatcher Proposition 4A.1 for a more explicit description of this action.

As a consequence of the exact sequence (2), we obtain a variant of Hatcher Proposition 4A.2.

Proposition 5.3. *Let (X, x_0) be a well-pointed space and (Y, y_0) any pointed space. Consider the natural map $\eta\colon [X, Y]_* \to [X, Y]$.*

1. *The map η is surjective if and only if Y is path-connected.*

2. *When Y is path-connected, η induces a bijection*

$$[X,Y]_*/\pi_1(Y) \xrightarrow{\cong} [X,Y]$$

where the left-hand side denotes the orbit set of $[X,Y]_*$ under the action of $\pi_1(Y)$.

3. *In particular, if Y is simply-connected, then η is a bijection.*

Corollary 5.4. *Let (X, x_0) be a well-pointed space, $n \geq 2$, and G an abelian group. Then the natural map*

$$[X, K(G,n)]_* \xrightarrow{\cong} [X, K(G,n)]$$

is a bijection.

If moreover X has the homotopy type of a CW complex, then both sides are naturally isomorphic to the cohomology group $H^n(X;G) \cong \widetilde{H}^n(X;G)$.

Proposition 5.5. *Let $n \geq 1$ and let G and H be groups (abelian if $n \geq 2$). Consider Eilenberg-MacLane spaces $K(G,n)$ and $K(H,n)$, where $K(G,n)$ is a CW complex. Then the map*

$$[K(G,n), K(H,n)]_* \xrightarrow{\pi_n} \mathrm{Hom}_{\mathbf{Gp}}(G,H)$$

is a bijection.

Proof. Hatcher § 4.3 Exercise 4.

When G and H are abelian, it follows from 2.3. Alternately, it can be shown directly with a model of $K(G,n)$ having generators of G as n-cells and relations as $(n+1)$-cells. In fact, the direct proof also proves the next statement. □

Proposition 5.6. *Let X be a path-connected CW complex and H a group. Then the map*

$$[X, K(H,1)]_* \xrightarrow{\pi_1} \mathrm{Hom}_{\mathbf{Gp}}\left(\pi_1(X), H\right)$$

is a bijection.

Proof. Homework 11 Problem 2. □

Proposition 5.7. *Let (X, x_0) be a well-pointed space and (Y, y_0) any pointed space. Then for any $n \geq 1$, the actions of $\pi_1(Y)$ on $[X,Y]_*$ and on $\pi_n(Y)$ are compatible. More precisely, the equation*

$$(\gamma \cdot f)_*(\alpha) = \gamma \cdot f_*(\alpha)$$

holds or all $\gamma \in \pi_1(Y)$, pointed map $f \colon X \to Y$, and $\alpha \in \pi_n(X)$. In other words, the diagram

$$\begin{tikzcd}
{[X,Y]_*} \arrow[r, "\pi_n"] \arrow[d, "\gamma\cdot-"'] & \mathrm{Hom}_{\mathbf{Gp}}\left(\pi_n(X), \pi_n(Y)\right) \arrow[d, "(\gamma\cdot-)_*"] \\
{[X,Y]_*} \arrow[r, "\pi_n"'] & \mathrm{Hom}_{\mathbf{Gp}}\left(\pi_n(X), \pi_n(Y)\right)
\end{tikzcd}$$

commutes.

Proof. Hatcher § 4.A Exercise 2. □

Corollary 5.8. *Let X be a path-connected CW complex and H a group. Then the map*

$$[X, K(H,1)] \xrightarrow{\pi_1} \mathrm{Hom}_{\mathbf{Gp}}(\pi_1(X), H)/\textit{conjugation in } H$$

is a bijection.

In particular, we have

$$[K(G,1), K(H,1)] \cong \mathrm{Hom}_{\mathbf{Gp}}(G, H)/\textit{conjugation in } H$$

Proof. Follows from 5.3, 5.6, 5.7, and the fact that the action of $\pi_1(K(H,1))$ on itself is by conjugation. □

Corollary 5.9. *Let X be a CW complex and G an abelian group. Then the map*

$$[X, K(G,1)]_* \xrightarrow{\cong} [X, K(G,1)]$$

is a bijection.

Moreover, both sides are naturally isomorphic to the cohomology group $H^1(X;G) \cong \widetilde{H}^1(X;G)$.

Proof. The case of X path-connected is proved like 5.8. The general case follows from the fact that X is the coproduct of its path components, along with the isomorphism

$$[X, K(G,1)]_* \cong [C_0, K(G,1)]_* \times \prod_{C \in \pi_0(X) \setminus \{C_0\}} [C, K(G,1)]$$

where $C_0 \in \pi_0(X)$ denotes the basepoint component. □

References

[1] John Milnor, *On spaces having the homotopy type of a* CW*-complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.

[2] James Stasheff, *A classification theorem for fibre spaces*, Topology **2** (1963), 239–246.