Math 527 - Homotopy Theory
Obstruction theory

Martin Frankland

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In our discussion of obstruction theory via the skeletal filtration, we left several claims as exercises. The goal of these notes is to fill in two of those gaps.

1 Setup

Let us recall the setup, adopting a notation similar to that of May § 18.5.

Let (X, A) be a relative CW complex with n-skeleton X_n, and let Y be a simple space. Given two maps $f_n, g_n \colon X_n \to Y$ which agree on X_{n-1}, we defined a **difference cochain**

$$d(f_n, g_n) \in C^n\left(X, A; \pi_n(Y)\right)$$

whose value on each n-cell was defined using the following "double cone construction".

Definition 1.1. Let $H, H' \colon D^n \to Y$ be two maps that agree on the boundary $\partial D^n \cong S^{n-1}$. The **difference construction** of H and H' is the map

$$H \cup H' \colon S^n \cong D^n \cup_{S^{n-1}} D^n \to Y.$$

Here, the two terms D^n are viewed as the upper and lower hemispheres of S^n respectively.

2 The two claims

In this section, we state two claims and reduce their proof to the case of spheres and discs.

Proposition 2.1. *Given two maps $f_n, g_n \colon X_n \to Y$ which agree on X_{n-1}, we have*

$$f_n \simeq g_n \ \textit{rel}\ X_{n-1}$$

if and only if $d(f_n, g_n) = 0$ holds.

Proof. For each n-cell e^n_α of $X \setminus A$, consider its attaching map $\varphi_\alpha \colon S^{n-1} \to X_{n-1}$ and characteristic map $\Phi_\alpha \colon (D^n, S^{n-1}) \to (X_n, X_{n-1})$.

Because f_n and g_n agree on X_{n-1}, the condition $f_n \simeq g_n$ rel X_{n-1} is equivalent to the corresponding condition on every n-cell:

$$f_n \circ \Phi_\alpha \simeq g_n \circ \Phi_\alpha \text{ rel } S^{n-1}.$$

By 3.3, this condition is equivalent to the condition

$$(f_n \circ \Phi_\alpha) \cup (g_n \circ \Phi_\alpha) = 0 \in \pi_n(Y)$$

for every n-cell, i.e. the vanishing of the difference cochain $d(f_n, g_n) = 0 \in C^n\left(X, A; \pi_n(Y)\right)$. □

Proposition 2.2. *Given a map $f_n \colon X_n \to Y$ and a cellular cochain $d \in C^n\left(X, A; \pi_n(Y)\right)$, there exists a map $g_n \colon X_n \to Y$ which agrees with f_n on X_{n-1}:*

$$f_n|_{X_{n-1}} = g_n|_{X_{n-1}}$$

and such that the difference cochain satisfies $d(f_n, g_n) = d$.

Proof. For each n-cell e^n_α of $X \setminus A$, consider its attaching map $\varphi_\alpha \colon S^{n-1} \to X_{n-1}$ and characteristic map $\Phi_\alpha \colon (D^n, S^{n-1}) \to (X_n, X_{n-1})$. To produce the desired map $g_n \colon X_n \to Y$, it suffices to define it on each n-cell of $X \setminus A$. The condition to be satisfied is that the difference construction

$$(f_n \circ \Phi_\alpha) \cup (g_n \circ \Phi_\alpha) \colon D^n \cup_{S^{n-1}} D^n \to Y$$

be a representative of the class $d(e^n_\alpha) \in \pi_n(Y)$.

This is always possible, by 3.4. □

3 The case of spheres and discs

The two propositions in the previous section boil down to properties of the difference construction, which we study in more detail.

Given maps $H, H' \colon D^n \to Y$ which agree on $\partial D^n \cong S^{n-1}$, it will be useful to think of H and H' as two null-homotopies of the same map

$$f := H|_{S^{n-1}} = H'|_{S^{n-1}} \colon S^{n-1} \to Y.$$

In that context, we view the disc as the cone on the sphere:

$$D^n \cong CS^{n-1} = S^{n-1} \times I / \big(S^{n-1} \times \{1\} \big) .$$

(Technically, we should take the reduced cone, but that's alright.)

Recall that for any pointed space X, the (reduced) suspension ΣX homotopy coacts on the (reduced) cone CX, via the map

$$c \colon CX \to CX \vee \Sigma X$$

which pinches the "middle" of the cone. Note moreover that this coaction map is compatible with the inclusions of X at the bottom of the cone:

$$\begin{array}{ccc} CX & \xrightarrow{\;c\;} & CX \vee \Sigma X \\ {\scriptstyle \iota_0} \uparrow & \nearrow {\scriptstyle \iota_0} & \\ X & & \end{array}$$

In particular, taking $X = S^{n-1}$, the sphere $S^n \cong \Sigma S^{n-1}$ homotopy coacts on the disc $D^n \cong CS^{n-1}$ via the coaction map

$$c \colon D^n \to D^n \vee S^n.$$

Notation 3.1. Let $f \colon S^{n-1} \to Y$ be a null-homotopic map. Denote by

$$[D^n, Y]_{f \text{ on } S^{n-1}}$$

the set of homotopy classes of maps $H \colon D^n \to Y$ rel S^{n-1} with restriction $H|_{S^{n-1}} = f \colon S^{n-1} \to Y$.

Precomposition by c yields an action of $\pi_n(Y)$ on $[D^n, Y]_{f \text{ on } S^{n-1}}$, which we denote by $\alpha \cdot H$. With appropriate sign conventions (namely that the pinch map $p \colon S^n \to S^n \vee S^n$ send the upper hemisphere to the first summand), this is a left action.

Proposition 3.2. *The difference construction satisfies the following properties.*

1. $H \cup H' = -H' \cup H$.

2. $(\alpha \cdot H) \cup H' = \alpha + (H \cup H')$ *for any* $\alpha \in \pi_n(Y)$.

3. $H \cup (\alpha \cdot H') = (H \cup H') - \alpha$ *for any* $\alpha \in \pi_n(Y)$.

Proof. 1. Using the model for the sphere $S^n \cong S^{n-1} \wedge S^1$, note that $H \cup H'$ and $H' \cup H$ differ by a flip of the last suspension coordinate:

$$\begin{array}{ccc} S^{n-1} \wedge S^1 \cong S^n & \xrightarrow{H \cup H'} & Y. \\ \downarrow{\scriptstyle \mathrm{id} \wedge (-1)} & \nearrow{\scriptstyle H' \cup H} & \\ S^{n-1} \wedge S^1 \cong S^n & & \end{array}$$

2. Straightforward (with an appropriate sign convention).

3. From 1 and 2, we conclude:

$$\begin{aligned} H \cup (\alpha \cdot H') &= -\left[(\alpha \cdot H') \cup H\right] \\ &= -\left[\alpha + (H' \cup H)\right] \\ &= -(H' \cup H) - \alpha \\ &= (H \cup H') - \alpha. \end{aligned}$$

Note that to cover the case $n = 1$, we allowed "addition" to be non-commutative. □

Proposition 3.3. *$H \simeq H'$ rel S^{n-1} holds if and only if $H \cup H' = 0 \in \pi_n(Y)$ holds.*

Proof. (⇒) A homotopy F from H to H' rel S^{n-1} defines a filler as illustrated here:

$$\begin{array}{ccc} S^n & \xrightarrow{H \cup H'} & Y \\ \big\downarrow & \nearrow{\scriptstyle F} & \\ D^{n+1} & & \end{array}$$

which proves $H \cup H' = 0 \in \pi_n(Y)$.

(⇐) Let us prove the relation

$$(H \cup H') \cdot H' \simeq H \text{ rel } S^{n-1}$$

from which we deduce the result:

$$\begin{aligned} H' &\simeq 0 \cdot H' \text{ rel } S^{n-1} \\ &\simeq (H \cup H') \cdot H' \text{ rel } S^{n-1} \\ &\simeq H \text{ rel } S^{n-1}. \end{aligned}$$

Up to rescaling, the map

$$(H \cup H') \cdot H' \colon D^n \cong S^{n-1} \times [0,3] / \big(S^{n-1} \times \{3\}\big) \to Y$$

is given by

$$(x,t) \mapsto \begin{cases} H'(x,t) & \text{if } 0 \le t \le 1 \\ H'(x, 2-t) & \text{if } 1 \le t \le 2 \\ H(x, t-2) & \text{if } 2 \le t \le 3. \end{cases}$$

The formula

$$(x,t,s) \mapsto \begin{cases} H'(x,st) & \text{if } 0 \le t \le 1 \\ H'(x, s(2-t)) & \text{if } 1 \le t \le 2 \\ H(x, t-2) & \text{if } 2 \le t \le 3 \end{cases}$$

for $s \in [0,1]$ provides a homotopy rel S^{n-1} between $(H \cup H') \cdot H'$ and a map which is clearly homotopic to H rel S^{n-1}. □

Proposition 3.4. *Given H as above and any $\alpha \in \pi_n(Y)$, there exists an H' satisfying $H \cup H' = \alpha \in \pi_n(Y)$.*

Proof. Take $H' = (-\alpha) \cdot H$. By 3.3, we have $H \cup H = 0 \in \pi_n(Y)$. By 3.2, we have the equality:

$$\begin{aligned} H \cup [(-\alpha) \cdot H] &= (H \cup H) - (-\alpha) \\ &= 0 + \alpha \\ &= \alpha. \end{aligned}$$

□

In fact, more is true.

Proposition 3.5. *The action of $\pi_n(Y)$ on the set $[D^n, Y]_{f \text{ on } S^{n-1}}$ is free and transitive.*

Proof. **Free.** Assume $\alpha \cdot H \simeq H$ rel S^{n-1} for $\alpha \in \pi_n(Y)$. By 3.3 and 3.2, we conclude:

$$\begin{aligned} &(\alpha \cdot H) \cup H = 0 \in \pi_n(Y) \\ =&\alpha + (H \cup H) \\ =&\alpha + 0 \\ =&\alpha. \end{aligned}$$

Transitive. Given two maps $H, H' \colon D^n \to Y$ satisfying $H|_{S^{n-1}} = H'|_{S^{n-1}} = f$, they are in the same $\pi_n(Y)$-orbit, by the relation

$$(H \cup H') \cdot H' \simeq H \text{ rel } S^{n-1}$$

which was proved in 3.3. □

Upshot. The difference construction $H \cup H'$ wants to be $H - H'$, but this does not make sense, because elements of $[D^n, Y]_{f \text{ on } S^{n-1}}$ cannot be added or subtracted. The next best thing is true: $[D^n, Y]_{f \text{ on } S^{n-1}}$ is a torsor for $\pi_n(Y)$, and $H \cup H' \in \pi_n(Y)$ is the *unique* element satisfying

$$(H \cup H') \cdot H' = H \in [D^n, Y]_{f \text{ on } S^{n-1}}.$$