Math 527 - Homotopy Theory
Spring 2013
Homework 12, Lecture 4/10

Problem 2. Let (X, e) be a pointed space. The **James construction** on X is the pointed space obtained by taking words in the elements of X and declaring that e is a unit. Formally, it is the quotient space:

$$J(X) := \coprod_{k \geq 1} X^k / \sim$$

where $\sim$ is the equivalence relation generated by identifications of the form:

$$(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_k) \sim (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k).$$

a. Show that $J(X)$ is a topological monoid (under concatenation of words).

b. Let M be a topological monoid and $f \colon X \to M$ a pointed map. Show that there is a unique continuous map of monoids $\widetilde{f} \colon J(X) \to M$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & M \\ {\scriptstyle \iota_1} \downarrow & \nearrow {\scriptstyle \widetilde{f}} & \\ J(X) & & \end{array}$$

commute. Here $\iota_1 \colon X \to J(X)$ denotes the canonical "inclusion of single-letter words", i.e. the composite

$$X = X^1 \hookrightarrow \coprod_{k \geq 1} X^k \twoheadrightarrow J(X).$$

Upshot. This shows that $J(X)$ is in fact the free topological monoid on X. In other words, let $U \colon \mathbf{TopMon} \to \mathbf{Top}_*$ denote the forgetful functor from topological monoids to pointed spaces. Then the functor $J \colon \mathbf{Top}_* \to \mathbf{TopMon}$ is left adjoint to U, and $\iota_1 \colon X \to J(X)$ is the unit map of the adjunction.