Math 527 - Homotopy Theory
Spring 2013
Homework 12, Lecture 4/12

Definition. Let (X, x_0) be a pointed space. The space of **Moore loops** $\Omega_M X$ in X is the space of pairs (γ, τ) with $\tau \in [0, \infty)$ and $\gamma \colon [0, \tau] \to X$ a loop at the basepoint, i.e. a continuous map satisfying $\gamma(0) = \gamma(\tau) = x_0$. It is topologized as the subspace:

$$\begin{aligned} \Omega_M X &= \{(\gamma, \tau) \in \operatorname{Map}([0,\infty), X) \times [0,\infty) \mid \gamma(0) = x_0 \text{ and } \gamma(t) = x_0 \text{ for all } t \geq \tau\} \\ &\subseteq \operatorname{Map}([0,\infty), X) \times [0,\infty). \end{aligned}$$

The basepoint of $\Omega_M X$ is the "instantaneous loop" $c_0 := (\gamma, 0)$.

Concatenation of Moore loops is defined as follows: $(\gamma_1, \tau_1) * (\gamma_2, \tau_2) \in \Omega_M X$ is the Moore loop $(\gamma, \tau_1 + \tau_2)$ given by

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } 0 \leq t \leq \tau_1 \\ \gamma_2(t - \tau_1) & \text{if } \tau_1 \leq t \leq \tau_1 + \tau_2 \end{cases}$$

also denoted $\gamma = \gamma_1 * \gamma_2$ by abuse of notation.

Concatenation makes $\Omega_M X$ into a (strict) monoid with unit c_0, and moreover one can check that it is a topological monoid, i.e. the concatenation map

$$* \colon \Omega_M X \times \Omega_M X \to \Omega_M X$$

is continuous.

Problem 3.

a. Show that the usual loop space ΩX and the Moore loop space $\Omega_M X$ are naturally homotopy equivalent, by an equivalence $\varphi \colon \Omega X \xrightarrow{\simeq} \Omega_M X$ which is moreover an H-map, i.e. such that the diagram

$$\begin{tikzcd} \Omega X \times \Omega X \arrow[r, "\varphi \times \varphi"] \arrow[d, "\text{concatenation}"'] & \Omega_M X \times \Omega_M X \arrow[d, "\text{concatenation}"] \\ \Omega X \arrow[r, "\varphi"'] & \Omega_M X \end{tikzcd}$$

commutes up to homotopy.

b. Deduce that the canonical map $\eta \colon X \to \Omega \Sigma X$ naturally extends up to homotopy to an H-map $\widetilde{\eta} \colon : J(X) \to \Omega \Sigma X$. Here $J(X)$ denotes the James construction on X (c.f. Problem 2).

"Extension up to homotopy" means that $\widetilde{\eta}$ makes the following diagram commute up to homotopy:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\iota_1"'] & \Omega \Sigma X. \\ J(X) \arrow[ur, "\widetilde{\eta}"'] & \end{tikzcd}$$