Math 527 - Homotopy Theory
Spring 2013
Homework 13, Lecture 4/17

Definition 1. A **reduced cohomology theory** is a family of (contravariant) functors

$$h^n \colon \mathbf{CW}_*^{\mathrm{op}} \to \mathbf{Ab}$$

for $n \in \mathbb{Z}$, from the category of pointed CW complexes to the category of abelian groups, satisfying the following axioms.

- **Homotopy invariance.** Homotopic maps $f \simeq g \colon X \to Y$ induce the same morphism $h^n(f) = h^n(g) \colon h^n(Y) \to h^n(X)$.
- **Exactness.** If $A \xrightarrow{i} X \xrightarrow{p} X/A$ is a cofiber sequence (say, $i \colon A \hookrightarrow X$ is the inclusion of a subcomplex), then the induced sequence

$$h^n(X/A) \xrightarrow{p^*} h^n(X) \xrightarrow{i^*} h^n(A)$$

  is exact. Moreover, there is a natural suspension isomorphism (which is part of the data of the cohomology theory):

$$h^n(X) \xrightarrow{\cong} h^{n+1}(\Sigma X).$$

- **Wedge axiom.** Each functor h^n sends wedges to products, i.e. the natural map

$$h^n\left(\bigvee_\alpha X_\alpha\right) \xrightarrow{\cong} \prod_\alpha h^n(X_\alpha)$$

  is an isomorphism.

Remark 2. In light of the iterated cofiber sequence

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \to \ldots$$

one could as well state the exactness axiom as a natural long exact sequence

$$\ldots \longrightarrow h^n(X/A) \xrightarrow{p^*} h^n(X) \xrightarrow{i^*} h^n(A) \xrightarrow{\delta} h^{n+1}(X/A) \longrightarrow \ldots$$

However, it is sometimes convenient to break up this information into two parts as we did above: exactness for each h^n, along with the suspension isomorphism relating successive functors h^n and h^{n+1}.

Remark 3. One could also view the functors $h^n \colon \mathbf{Top}_* \to \mathbf{Ab}$ as defined on the category of pointed spaces, and require that they be weak homotopy invariant. That is, if $f \colon X \xrightarrow{\sim} Y$ is a weak homotopy equivalence, then $h^n(f) \colon h^n(Y) \xrightarrow{\cong} h^n(X)$ is an isomorphism.

Definition 4. An **Ω-spectrum** (sometimes called simply a spectrum) E is a family of pointed spaces $\{E_n\}_{n\in\mathbb{N}}$ endowed with structure maps

$$\omega_n \colon E_n \xrightarrow{\sim} \Omega E_{n+1}$$

which are weak homotopy equivalences, for all $n \in \mathbb{N} = \{0, 1, \ldots\}$.

Remark 5. It is customary to require that each E_n have the homotopy type of a CW complex, and in particular the structure maps $E_n \xrightarrow{\simeq} \Omega E_{n+1}$ are homotopy equivalences. This is important when using spectra to describe homology theories, c.f. Hatcher § 4.F and May § 22.1. When using spectra to describe cohomology theories, as we will do below, that requirement is not needed, and having weak homotopy equivalences $E_n \xrightarrow{\sim} \Omega E_{n+1}$ is good enough.

Remark 6. One can always view a spectrum as indexed over $\mathbb{Z}$ instead of $\mathbb{N}$, by letting $E_{-m} := \Omega^m E_0$ for $m > 0$, with identity structure maps $\Omega^m E_0 \xrightarrow{=} \Omega(\Omega^{m-1} E_0)$. These iterated loop spaces provide no additional information. The information in an Ω-spectrum is contained in the successive **de**loopings of E_0, i.e. what happens as $n \to +\infty$.

Problem 3. Let $E = \{E_n\}_{n\in\mathbb{N}}$ be an Ω-spectrum. Show that the assignments

$$h^n(X) := [X, E_n]_*$$

define a reduced cohomology theory $\{h^n\}_{n\in\mathbb{Z}}$. Don't forget to address the abelian group structure of $h^n(X)$.

Here we use the convention described in Remark 6 for $n < 0$.

Problem 4. Let $h^* = \{h^n\}_{n\in\mathbb{Z}}$ be a reduced cohomology theory. Show that there is an Ω-spectrum E representing h^* in the sense of Problem 3. Explicitly: there are natural isomorphisms of abelian groups

$$h^n(X) \cong [X, E_n]_*$$

for all $n \in \mathbb{Z}$ which are moreover compatible with the suspension isomorphisms, i.e. making the diagram:

$$\begin{array}{ccc} h^n(X) & \xrightarrow{\cong} & [X, E_n]_* \\ \cong\downarrow & & \downarrow\cong \\ h^{n+1}(\Sigma X) & \xrightarrow[\cong]{} & [\Sigma X, E_{n+1}]_* \end{array}$$

commute.