Math 527 - Homotopy Theory
Spring 2013
Homework 6, Lecture 2/18

Problem 1. (The Hopf fibration) Let $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ be the unit sphere. Stereographic projection provides a homeomorphism $S^2 \cong \mathbb{C}P^1$, where the "North pole" corresponds to $[0 : 1] \in \mathbb{C}P^1$. The composite

$$S^3 \hookrightarrow \mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{C}P^1$$

where the second map is the natural quotient map, is called the **Hopf map** and is usually denoted by $\eta \colon S^3 \to S^2$.

a. Show that $\eta \colon S^3 \to S^2$ is a fiber bundle with fiber S^1.

b. Show that for all $n \geq 3$, the Hopf map induces an isomorphism $\eta_* \colon \pi_n(S^3) \xrightarrow{\simeq} \pi_n(S^2)$. Deduce in particular the isomorphism $\pi_3(S^2) \simeq \mathbb{Z}$, where the class $[\eta] \in \pi_3(S^2)$ is a generator.