Math 535 - General Topology
Additional notes

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1 Subspaces

Definition 1.1. Let X be a topological space and $A \subseteq X$ any subset. The **subspace topology** on A is the smallest topology $\mathcal{T}_A^{\text{sub}}$ making the inclusion map $i\colon A \hookrightarrow X$ continuous.

In other words, $\mathcal{T}_A^{\text{sub}}$ is generated by subsets $V \subseteq A$ of the form

$$V = i^{-1}(U) = U \cap A$$

for any open $U \subseteq X$.

Proposition 1.2. *The subspace topology on A is*

$$\mathcal{T}_A^{sub} = \{V \subseteq A \mid V = U \cap A \text{ for some open } U \subseteq X\}.$$

In other words, the collection of subsets of the form $U \cap A$ already forms a topology on A.

2 Products

Before discussing the product of spaces, let us review the notion of product of sets.

2.1 Product of sets

Let X and Y be sets. The Cartesian product of X and Y is the set of pairs

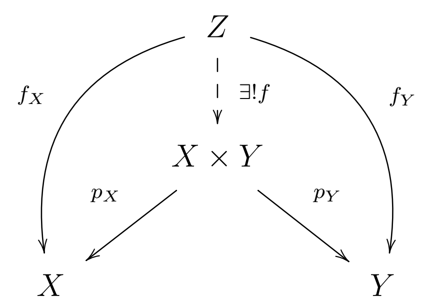
$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

It comes equipped with the two projection maps $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ onto each factor, defined by

$$p_X(x, y) = x$$
$$p_Y(x, y) = y.$$

This explicit description of $X \times Y$ is made more meaningful by the following proposition.

Proposition 2.1. *The Cartesian product of sets satisfies the following universal property. For any set Z along with maps $f_X\colon Z \to X$ and $f_Y\colon Z \to Y$, there is a unique map $f\colon Z \to X \times Y$ satisfying $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$, in other words making the diagram*



commute.

Proof. Given f_X and f_Y, define $f\colon Z \to X \times Y$ by

$$f(z) := (f_X(z), f_Y(z))$$

which clearly satisfies $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$.

To prove uniqueness, note that any pair $(x, y) \in X \times Y$ can be written as

$$(x, y) = (p_X(x, y), p_Y(x, y))$$

i.e. the projections give us each individual component of the pair. Therefore, any function $g\colon Z \to X \times Y$ can be written as

$$\begin{aligned} g(z) &= (p_X(g(z)), p_Y(g(z))) \\ &= ((p_X \circ g)(z), (p_Y \circ g)(z)) \end{aligned}$$

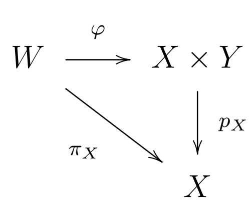
so that g is *determined* by its components $p_X \circ g$ and $p_Y \circ g$. $\square$

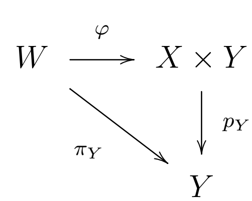
In slogans: "A map into $X \times Y$ is the same data as a map into X and a map into Y".

Yet another slogan: "$X \times Y$ is the closest set equipped with a map to X and a map to Y."

As usual with universal properties, this characterizes $X \times Y$ up to unique isomorphism. This statement is made precise in the following proposition.

Proposition 2.2. *Let W be a set equipped with maps $\pi_X\colon W \to X$ and $\pi_Y\colon W \to Y$ satisfying the universal property of the product. Then there is a unique isomorphism $\varphi\colon W \xrightarrow{\cong} X \times Y$ commuting with the projections, i.e. making the diagrams*





commute.

Proof. Starting from the data of the maps $\pi_X \colon W \to X$ and $\pi_Y \colon W \to Y$, the universal property of $X \times Y$ provides a unique map $\varphi \colon W \to X \times Y$ commuting with the projections.

Likewise, starting from the data of the maps $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$, the universal property of W provides a unique map $\psi \colon X \times Y \to W$ commuting with the projections.

We claim that φ is an isomorphism, with inverse ψ.

The composite $\psi \circ \varphi \colon W \to W$ is a map into W commuting with the projections. But so is the identity map $\mathrm{id}_W \colon W \to W$. By uniqueness (guaranteed in the universal property of W), we obtain $\psi \circ \varphi = \mathrm{id}_W$.

Likewise, the composite $\varphi \circ \psi \colon X \times Y \to X \times Y$ is a map into $X \times Y$ commuting with the projections. But so is the identity map $\mathrm{id}_{X \times Y} \colon X \times Y \to X \times Y$. By uniqueness (guaranteed in the universal property of $X \times Y$), we obtain $\varphi \circ \psi = \mathrm{id}_{X \times Y}$. □

2.2 Product topology

The next goal is to define the product $X \times Y$ of topological spaces X and Y such that it satisfies the analogous universal property in the category of topological spaces.

In other words, we want to find a topology on $X \times Y$ such that the projection maps $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ are *continuous*, and such that for any topological space Z along with *continuous* maps $f_X \colon Z \to X$ and $f_Y \colon Z \to Y$, there is a unique *continuous* map $f \colon Z \to X \times Y$ satisfying $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$.

Definition 2.3. Let X and Y be topological spaces. The **product topology** $\mathcal{T}_{X \times Y}$ on $X \times Y$ is the smallest topology on $X \times Y$ making the projections $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ continuous.

In other words, $\mathcal{T}_{X \times Y}$ is generated by "strips" of the form

$$p_X^{-1}(U) = U \times Y$$
$$p_Y^{-1}(V) = X \times V$$

for some open $U \subseteq X$ or some open $V \subseteq Y$.

Proposition 2.4. *The collection of "rectangles"*

$$\{U \times V \mid U \subseteq X \text{ is open and } V \subseteq Y \text{ is open}\}$$

is a basis for the product topology on $X \times Y$.

Proof. Finite intersections of strips

$$(U \times Y) \cap (X \times V) = U \times V$$

provide all rectangles. However a finite intersection of rectangles

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$$

is again a rectangle, since $U_1 \cap U_2 \subseteq X$ is open and $V_1 \cap V_2 \subseteq Y$ is open. □

Proposition 2.5. *The topological space $(X \times Y, \mathcal{T}_{X \times Y})$ along with the projections $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ satisfies the universal property of a product.*

Proof. Let Z be a topological space along with continuous maps $f_X \colon Z \to X$ and $f_Y \colon Z \to Y$. In particular, these continuous maps are functions, so that there is a unique function $f \colon Z \to X \times Y$ satisfying $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$. In other words, f is given by

$$f(z) = (f_X(z), f_Y(z)) .$$

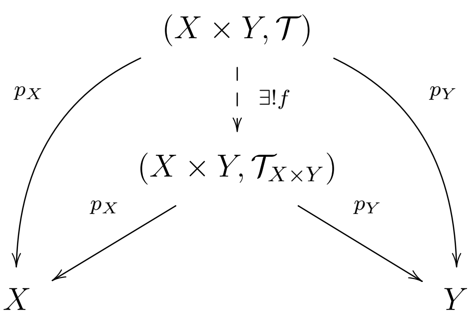
It remains to check that f is continuous. For any rectangle $U \times V \subseteq X \times Y$ where $U \subseteq X$ is open and $V \subseteq Y$ is open, its preimage is

$$\begin{aligned} f^{-1}(U \times V) &= \{z \in Z \mid f(z) \in U \times V\} \\ &= \{z \in Z \mid f_X(z) \in U \text{ and } f_Y(z) \in V\} \\ &= f_X^{-1}(U) \cap f_Y^{-1}(V). \end{aligned}$$

Since f_X and f_Y are continuous, the subsets $f_X^{-1}(U)$ and $f_Y^{-1}(V)$ are open in Z, and so is their intersection $f_X^{-1}(U) \cap f_Y^{-1}(V)$. Since those rectangles $U \times V$ form a basis for the product topology on $X \times Y$, the function $f \colon Z \to X \times Y$ is continuous. □

Remark 2.6. Why did we choose the *smallest* topology making the projections p_X and p_Y continuous?

If there is a product topology $\mathcal{T}_{X \times Y}$ satisfying the universal property, consider any other topology $\mathcal{T}$ on $X \times Y$ making the projections p_X and p_Y continuous. Then the universal property of $\mathcal{T}_{X \times Y}$ provides a unique *continuous* map f making the diagram



commute. As a function, $f \colon X \times Y \to X \times Y$ must be the identity:

$$\begin{aligned} f(x, y) &= (p_X(x, y), p_Y(x, y)) \\ &= (x, y). \end{aligned}$$

The identity $\mathrm{id} \colon (X \times Y, \mathcal{T}) \to (X \times Y, \mathcal{T}_{X \times Y})$ being continuous means precisely the inequality $\mathcal{T}_{X \times Y} \leq \mathcal{T}$. That is why $\mathcal{T}_{X \times Y}$ had to be the *smallest* topology making the projections continuous.

Exercise 2.7. Let (X, d_X) and (Y, d_Y) be metric spaces.

1. For points (x, y) and (x', y') in $X \times Y$, define their distance as the sum

$$d\left((x, y), (x', y')\right) := d_X(x, x') + d_Y(y, y').$$

   Show that d is a metric on $X \times Y$.

2. Show that the metric d induces the product topology on $X \times Y$.