Math 535 - General Topology
Additional notes

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1 Infinite products

Definition 1.1. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The **product topology** $\mathcal{T}_{\text{prod}}$ on the Cartesian product $\prod_\alpha X_\alpha$ is the smallest topology making all projection maps $p_\beta \colon \prod_\alpha X_\alpha \to X_\beta$ continuous.

In other words, the product topology is generated by subsets of the form $p_\beta^{-1}(U_\beta)$ for $U_\beta \subseteq X_\beta$ open.

A basis for $\mathcal{T}_{\text{prod}}$ is the collection of "large boxes"

$$\{\prod_\alpha U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open, and } U_\alpha = X_\alpha \text{ except for at most finitely many } \alpha\}$$

Proposition 1.2. *The topological space $(\prod_\alpha X_\alpha, \mathcal{T}_{prod})$ along with the projections $p_\beta \colon \prod_\alpha X_\alpha \to X_\beta$ satisfies the universal property of a product.*

Proof. Let Z be a topological space along with continuous maps $f_\alpha \colon Z \to X_\alpha$ for all $\alpha \in A$. In particular, these continuous maps are functions, so that there is a unique function $f \colon Z \to \prod_\alpha X_\alpha$ whose components are $p_\alpha \circ f = f_\alpha$. In other words, f is given by

$$f(z) = (f_\alpha(z))_{\alpha \in A}.$$

It remains to check that f is continuous. The product topology is generated by subsets of the form $p_\beta^{-1}(U_\beta)$ for $U_\beta \subseteq X_\beta$ open. Its preimage under f is

$$\begin{aligned} f^{-1}\left(p_\beta^{-1}(U_\beta)\right) &= (p_\beta \circ f)^{-1}(U_\beta) \\ &= f_\beta^{-1}(U_\beta) \end{aligned}$$

which is open in Z since $f_\beta \colon Z \to X_\beta$ is continuous. □

Definition 1.3. The **box topology** $\mathcal{T}_{\text{box}}$ on the Cartesian product $\prod_\alpha X_\alpha$ is the topology for which the collection of "boxes"

$$\{\prod_\alpha U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open}\}$$

is a basis.

Note that we always have $\mathcal{T}_{\text{prod}} \leq \mathcal{T}_{\text{box}}$, and equality holds for finite products. For an infinite product, the inequality is usually strict.

Exercise 1.4. Show that the projection maps $p_\beta \colon \prod_\alpha X_\alpha \to X_\beta$ are open maps in the box topology (and therefore also in the product topology).

2 Disjoint unions

In this section, we describe a construction which is dual to the product. The discussion will be eerily similar to that of products, because the ideas are the same, and because of copy-paste.

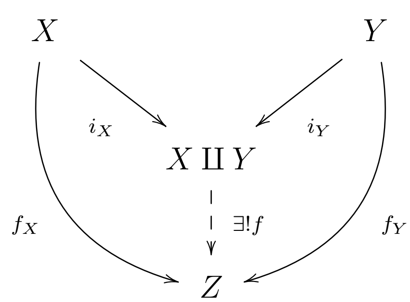
2.1 Disjoint union of sets

Let X and Y be sets. The disjoint union of X and Y is the set

$$X \amalg Y = \{w \mid w \in X \text{ or } x \in Y\}.$$

It comes equipped with the inclusion maps $i_X \colon X \to X \amalg Y$ and $i_Y \colon Y \to X \amalg Y$ from each summand. This explicit description of $X \amalg Y$ is made more meaningful by the following proposition.

Proposition 2.1. *The disjoint union of sets $X \amalg Y$, along with inclusion maps i_X and i_Y, is the **coproduct** of sets, i.e. it satisfies the following universal property. For any set Z along with maps $f_X \colon X \to Z$ and $f_Y \colon Y \to Z$, there is a unique map $f \colon X \amalg Y \to Z$ whose restrictions are $f \circ i_X = f_X$ and $f \circ i_Y = f_Y$, in other words making the diagram*



commute.

Proof. Given f_X and f_Y, define $f \colon X \amalg Y \to Z$ by

$$f(w) := \begin{cases} f_X(w) & \text{if } w \in X \\ f_Y(w) & \text{if } w \in Y \end{cases}$$

which clearly satisfies $f \circ i_X = f_X$ and $f \circ i_Y = f_Y$.

To prove uniqueness, note that any element $w \in X \amalg Y$ is in one of the summands:

$$w = \begin{cases} i_X(w) & \text{if } w \in X \\ i_Y(w) & \text{if } w \in Y. \end{cases}$$

Therefore, any function $g\colon X \amalg Y \to Z$ can be written as

$$g(w) = \begin{cases} g(i_X(w)) = (g \circ i_X)(w) & \text{if } w \in X \\ g(i_Y(w)) = (g \circ i_Y)(w) & \text{if } w \in Y \end{cases}$$

so that g is *determined* by its restrictions $g \circ i_X$ and $g \circ i_Y$. □

In slogans: "A map out of $X \amalg Y$ is the same data as a map out of X and a map out of Y".

Yet another slogan: "$X \amalg Y$ is the closest set equipped with a map from X and a map from Y."

As usual with universal properties, this characterizes $X \amalg Y$ up to unique isomorphism.

2.2 Coproduct topology

The next goal is to define the coproduct $X \amalg Y$ of topological spaces X and Y such that it satisfies the analogous universal property in the category of topological spaces.

In other words, we want to find a topology on $X \amalg Y$ such that the inclusion maps $i_X\colon X \to X \amalg Y$ and $i_Y\colon Y \to X \amalg Y$ are *continuous*, and such that for any topological space Z along with *continuous* maps $f_X\colon X \to Z$ and $f_Y\colon Y \to Z$, there is a unique *continuous* map $f\colon X \amalg Y \to Z$ whose restrictions are $f \circ i_X = f_X$ and $f \circ i_Y = f_Y$.

Definition 2.2. Let X and Y be topological spaces. The **coproduct topology** is the largest topology on $X \amalg Y$ making the inclusions $i_X\colon X \to X \amalg Y$ and $i_Y\colon Y \to X \amalg Y$ continuous.

This means that a subset $U \subseteq X \amalg Y$ is open if and only if $i_X^{-1}(U)$ is open in X and $i_Y^{-1}(U)$ is open in Y.

More concretely, noting $i_X^{-1}(U) = U \cap X$ and $i_Y^{-1}(U) = U \cap Y$, open sets can be described as $U = U_X \amalg U_Y$ where $U_X = U \cap X$ is open in X and $U_Y = U \cap Y$ is open in Y.

This definition works for an infinite disjoint union as well.

Definition 2.3. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. The **coproduct topology** $\mathcal{T}_{\text{coprod}}$ on the disjoint union $\coprod_\alpha X_\alpha$ is the largest topology making all inclusion maps $i_\beta\colon X_\beta \to \coprod_\alpha X_\alpha$ continuous.

This means that a subset $U \subseteq \coprod_\alpha X_\alpha$ is open if and only if $i_\alpha^{-1}(U)$ is open in X_α for all $\alpha \in A$.

More concretely, noting $i_\alpha^{-1}(U) = U \cap X_\alpha$, open sets can be described as $U = \coprod_\alpha U_\alpha$ where $U_\alpha = U \cap X_\alpha$ is open in X_α. That is, open subsets are disjoint unions of open subsets from each of the summands.

Proposition 2.4. *Each summand $X_\beta \subseteq \coprod_\alpha X_\alpha$ is open in the coproduct topology.*

Proof. Write $X_\beta = \coprod_\alpha U_\alpha$ where

$$U_\alpha = \begin{cases} X_\beta & \text{if } \alpha = \beta \\ \emptyset & \text{if } \alpha \neq \beta \end{cases}$$

is open in X_α for all α. □

Remark 2.5. More generally, the same proof shows that each inclusion map $i_\beta \colon X_\beta \to \coprod_\alpha X_\alpha$ is an open map.

Proposition 2.6. *The topological space* $(\coprod_\alpha X_\alpha, \mathcal{T}_{coprod})$ *along with the inclusions* $i_\beta \colon X_\beta \to \coprod_\alpha X_\alpha$ *is a coproduct of topological spaces.*

Proof. We verify the universal property of a coproduct.

Let Z be a topological space along with continuous maps $f_\alpha \colon X_\alpha \to Z$ for all $\alpha \in A$. In particular, these continuous maps are functions, so that there is a unique function $f \colon \coprod_\alpha X_\alpha \to Z$ whose restrictions are $f \circ i_\alpha = f_\alpha$. In other words, f is given by

$$f(w) = f(i_\alpha(w)) = f_\alpha(w)$$

where α is the unique index for which $w \in X_\alpha$.

It remains to check that f is continuous. Let $U \subseteq Z$ be open and consider its preimage $f^{-1}(U) \subseteq \coprod_\alpha X_\alpha$. To show that this subset is open, it suffices to check that its restriction to each summand is open:

$$\begin{aligned} i_\alpha^{-1}\left(f^{-1}(U)\right) &= (f \circ i_\alpha)^{-1}(U) \\ &= f_\alpha^{-1}(U) \end{aligned}$$

is indeed open in X_α since $f_\alpha \colon X_\alpha \to Z$ is continuous. □

Upshot: A map $f \colon \coprod_\alpha X_\alpha \to Z$ is continuous if and only if its restriction $f \circ i_\alpha \colon X_\alpha \to Z$ to each summand is continuous.