Math 535 - General Topology
Additional notes

Martin Frankland

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1 Quotient spaces

Definition 1.1. Let X be a topological space and $\sim$ an equivalence relation on X, along with the canonical projection $\pi \colon X \twoheadrightarrow X/\sim$. The **quotient topology** on $X/\sim$ is the largest topology making π continuous.

Explicitly, a subset $U \subseteq X/\sim$ is open if and only if its preimage $\pi^{-1}(U) \subseteq X$ is open in X.

Proposition 1.2. *With the quotient topology on $X/\sim$, a map $g \colon X/\sim \to Z$ is continuous if and only if the composite $g \circ \pi \colon X \to Z$ is continuous.*

Proof. Homework 2 Problem 5. □

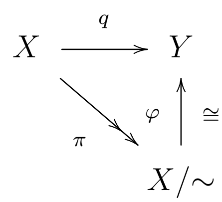
Proposition 1.3. *The space $X/\sim$ endowed with the quotient topology satisfies the universal property of a quotient. More precisely, the projection $\pi \colon X \twoheadrightarrow X/\sim$ is continuous, and for any continuous map $f \colon X \to Z$ which is constant on equivalence classes, there is a unique continuous map $\overline{f} \colon X/\sim \to Z$ such that $f = \overline{f} \circ \pi$, i.e. making the diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle\pi}\downarrow & \nearrow_{\exists!\overline{f}} & \\ X/\sim & & \end{array}$$

commute.

Proof. By the universal property of the projection map in sets, there is a unique function $\overline{f} \colon X/\sim \to Z$ such that $f = \overline{f} \circ \pi$. It remains to check that $\overline{f}$ is continuous. By proposition 1.2, the fact that $\overline{f} \circ \pi$ is continuous guarantees that $\overline{f}$ is continuous. □

Definition 1.4. Let X and Y be topological spaces. A map $q \colon X \to Y$ is called a **quotient map** or **identification map** if it is, up to homeomorphism, of the form $\pi \colon X \twoheadrightarrow X/\sim$ where $X/\sim$ is endowed with the quotient topology. More precisely, q is a quotient map if there exists an equivalence relation $\sim$ on X and a homeomorphism $\varphi\colon X/{\sim} \xrightarrow{\cong} Y$ making the diagram



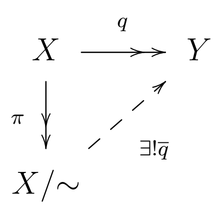
commute.

Note that the definition implies that q must be continuous and surjective, and that the equivalence relation $\sim$ on X must be the one induced by q, namely $x \sim x'$ if and only if $q(x) = q(x')$.

How to recognize quotient maps? In sets, a quotient map is the same as a surjection. However, in topological spaces, being continuous and surjective is not enough to be a quotient map. The crucial property of a quotient map is that open sets $U \subseteq X/{\sim}$ can be "detected" by looking at their preimage $\pi^{-1}(U) \subseteq X$.

Proposition 1.5. *Let $q\colon X \twoheadrightarrow Y$ be a surjective continuous map satisfying that $U \subseteq Y$ is open if and only if its preimage $q^{-1}(U) \subseteq X$ is open. Then q is a quotient map.*

Proof. Let $\sim$ be the equivalence relation on X induced by q, i.e. $x \sim x'$ if and only if $q(x) = q(x')$. By definition, $q\colon X \to Y$ is constant on equivalence classes. By the universal property of the quotient space $X/{\sim}$, there is a unique continuous map $\overline{q}\colon X/{\sim} \to Y$ such that $\overline{q} \circ \pi = q$, i.e. making the diagram



commute. By construction, $\overline{q}$ is now bijective. To prove that it is a homeomorphism, it remains to show that it is an open map.

Let $U \subseteq X/{\sim}$ be open. We want to show that $\overline{q}(U) \subseteq Y$ is open. By assumption, q has the property of "detecting" open subsets of Y, i.e. it suffices to check that the preimage $q^{-1}(\overline{q}(U)) \subseteq X$ is open. This preimage is

$$\begin{aligned} q^{-1}(\overline{q}(U)) &= (\overline{q} \circ \pi)^{-1}(\overline{q}(U)) \\ &= \pi^{-1}\overline{q}^{-1}(\overline{q}(U)) \\ &= \pi^{-1}(U) \text{ since } \overline{q} \text{ is injective} \end{aligned}$$

which is open in X since $\pi\colon X \twoheadrightarrow X/{\sim}$ is continuous. □