Math 535 - General Topology
Additional notes

Martin Frankland

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1 Hausdorff spaces

Definition 1.1. A topological space X is **Hausdorff** (or T_2) if for any distinct points $x, y \in X$, there exist open subsets $U, V \subset X$ satisfying $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

In other words, distinct points can be separated by neighborhoods.

Example 1.2. Every metric space is Hausdorff.

Proposition 1.3 (Uniqueness of limits). *Let X be a Hausdorff topological space, and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X with $x_n \to x$ and $x_n \to x'$. Then $x = x'$.*

In other words: Limits of sequences are unique (if they exist).

Proposition 1.4.

1. *A subspace of a Hausdorff space is Hausdorff.*
2. *An arbitrary product of Hausdorff spaces is Hausdorff.*

Remark 1.5. Quotients of Hausdorff spaces need not be Hausdorff.

Example 1.6 (Line with two origins). Consider the disjoint union $\mathbb{R} \coprod \mathbb{R}$, where we write $(t, 1)$ for elements in the first summand and $(t, 2)$ in the second summand, $t \in \mathbb{R}$. Let $X = (\mathbb{R} \coprod \mathbb{R})/\sim$ where the equivalence relation is generated by $(t, 1) \sim (t, 2)$ for all $t \neq 0$. In other words, we glue together the two lines everywhere except at the origin.

This space X is not Hausdorff, because the two distinct origins $(0, 1)$ and $(0, 2)$ cannot be separated by neighborhoods. For any open neighborhoods U of $(0, 1)$ and V of $(0, 2)$ in X, we have $U \cap V \neq \emptyset$.

2 Countability axioms

2.1 First-countable

Definition 2.1. Let X be a topological space. A **neighborhood basis** for a point $x \in X$ is a collection $\mathcal{B}_x$ of neighborhoods of x such that for any neighborhood N of x, there is some $B \in \mathcal{B}$ satisfying $B \subseteq N$.

Definition 2.2. A topological space X is **first-countable** if every point $x \in X$ has a countable neighborhood basis.

Example 2.3. Every metric space is first-countable. For $x \in X$, consider the neighborhood basis

$$\mathcal{B}_x = \{B_r(x) \mid r > 0, r \in \mathbb{Q}\}$$

consisting of open balls around x of rational radius.

Proposition 2.4. *Let X be a first-countable topological space, and $A \subseteq X$ a subset. Let $x \in \overline{A}$ be in the closure of A. Then there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A satisfying $a_n \to x$.*

Example 2.5. The space $\mathbb{R}^{\mathbb{N}}$ with the *box* topology is not first-countable. Indeed, we found a subset $A = \{x \in \mathbb{R}^{\mathbb{N}} \mid x_n > 0 \text{ for all } n \in \mathbb{N}\}$ and a point $\underline{0} = (0, 0, 0, \ldots) \in \overline{A}$ which is not the limit of any sequence in A.

Corollary 2.6. *Let X be a first-countable topological space.*

1. *A subset $C \subseteq X$ is closed if and only if whenever a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C satisfies $x_n \to x$, then we have $x \in C$.*

2. *A subset $U \subseteq X$ is open if and only if whenever a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X satisfies $x_n \to x \in U$, then the sequence is eventually in U.*

Proposition 2.7. *Let X and Y be topological spaces, where X is first-countable. A map $f\colon X \to Y$ is continuous at $x \in X$ if and only if whenever $x_n \to x$, we have $f(x_n) \to f(x)$.*

Proof. ($\Rightarrow$) Always true for any topological spaces.

($\Leftarrow$) Assume f is discontinuous at $x \in X$, which means there is a neighborhood N of $f(x)$ such that for any neighborhood M of x, we have $f(M) \not\subseteq N$. Since X is first-countable, there is a countable neighborhood basis $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ of x. Because of the condition $f(M_i) \not\subseteq N$, we can pick $x_i \in M_i$ such that $f(x_i) \notin N$.

Then the sequence $\{x_i\}_{i\in\mathbb{N}}$ satisfies $x_i \to x$ but $f(x_i)$ is never in N, so in particular $f(x_i) \not\to f(x)$. □

In other words, continuity always implies sequential continuity, but if the domain X is first-countable, then continuity is equivalent to sequential continuity.

Proposition 2.8.
1. *A subspace of a first-countable space is first-countable.*

2. *A* countable *product of first-countable spaces is first-countable.*

2.2 Second-countable

Definition 2.9. A topological space X is **second-countable** if its topology has a countable basis.

Example 2.10. Euclidean space $\mathbb{R}^n$ is second-countable, because the collection

$$\mathcal{B} = \{B_r(x) \mid x \in \mathbb{Q}^n, r > 0, r \in \mathbb{Q}\}$$

consisting of open balls of rational radius around points with rational coordinates is a basis for the topology, and $\mathcal{B}$ is a countable collection.

Proposition 2.11. *A second-countable space is always first-countable.*

Proof. Let $\mathcal{B}$ be a countable basis for the topology of X, and let $x \in X$. Then the collection

$$\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in B\}$$

is a neighborhood basis for x, and it is countable. □

Remark 2.12. The converse does not hold. For example, consider X an uncountable set endowed with the discrete topology. Then X is first-countable but not second-countable.

Remark 2.13. We have seen that a metric space is always first-countable. However, it need not be second-countable. For example, consider again X an uncountable set endowed with the discrete topology. Then X is metrizable but not second-countable.

In diagrams: