Math 535 - General Topology
Additional notes

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1 Compactness

1.1 Definitions

Definition 1.1. Let X be a topological space.

- A **cover** of X is a collection $\{U_\alpha\}_{\alpha \in A}$ of subsets $U_\alpha \subseteq X$ satisfying $X = \bigcup_{\alpha \in A} U_\alpha$.
- An **open cover** of X is a cover $\{U_\alpha\}_{\alpha \in A}$ where each U_α is open in X.
- A **subcover** of $\{U_\alpha\}_{\alpha \in A}$ is a subcollection $\{U_\beta\}_{\beta \in B}$ (for some $B \subseteq A$) which is still a cover, i.e. $X = \bigcup_{\beta \in B} U_\beta$.

Definition 1.2. A topological space X is **compact** if for every open cover $\{U_\alpha\}_{\alpha \in A}$ of X, there is a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$, i.e. $X = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.

1.2 Facts about compactness

Proposition 1.3. *Let X be a topological space and $Y \subseteq X$ a subspace. Then Y is compact if and only if for every collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets $U_\alpha \subseteq X$ satisfying $Y \subseteq \bigcup_{\alpha \in A} U_\alpha$, there is a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ satisfying $Y \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$.*

Proposition 1.4. *Let $K_1, \ldots, K_n$ be compact subspaces of X. Then their union $K_1 \cup \ldots \cup K_n$ is compact.*

Slogan: "Finite union of compact is compact".

Proposition 1.5. *Let $f \colon X \to Y$ be a continuous map between topological spaces, and assume X is compact. Then $f(X)$ is compact.*

Slogan: "Continuous image of compact is compact".

Remark 1.6. In particular, a quotient of a compact space is always compact.

Proposition 1.7. *Let X be a compact topological space and $C \subseteq X$ a closed subspace. Then C is compact.*

Slogan: "closed in compact is compact".

Proposition 1.8. *Let X be a Hausdorff topological space and $K \subseteq X$ a compact subspace. Then K is closed in X.*

Slogan: "compact inside Hausdorff is closed".

Example 1.9. Let X be an anti-discrete space. Then every subspace $Y \subset X$ is compact, though most of them are not closed in X (only the empty set $\emptyset$ and X itself are closed in X).

Proposition 1.10. *Let $f\colon X \to Y$ be a continuous map between topological spaces, where X is compact and Y a Hausdorff. Then f is a closed map.*

In particular, if f is a continuous bijection, then f is a homeomorphism.

1.3 An important example

A basic example of compact space, yet one of the most important, is provided by the following classic theorem.

Theorem 1.11 (Bolzano-Weierstrass)**.** *The interval $[0,1]$ is compact.*

Proof. Suppose $[0,1]$ is not compact, i.e. there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ which does not admit a finite subcover. Then either $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ (or both) cannot be covered by a finite subcover. Call this new interval $[a_1,b_1]$, where we write $[a_0,b_0] := [0,1]$.

Repeating the argument, for every $n \geq 0$, we obtain an interval $[a_n,b_n]$ which cannot be covered by a finite subcover, and each interval has length $b_n - a_n = \frac{1}{2^n}$. Moreover, the intervals are nested (decreasing):

$$[a_0,b_0] \supset [a_1,b_1] \supset [a_2,b_2] \supset \dots$$

The sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are monotone and bounded, therefore they converge, say $a_n \to a$ and $b_n \to b$. We have

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}\frac{1}{2^n} = b - a = 0$$

so that $a = b$. This point $a \in [0,1]$ is in some U_{α_0}, which is open, so we can find some small radius $\epsilon > 0$ such that the open ball $(a-\epsilon, a+\epsilon) \subseteq U_{\alpha_0}$. (To be nitpicky, we should instead write $(a-\epsilon,a+\epsilon)\cap[0,1]$, which is an open ball in $[0,1]$.)

By the convergence $a_n \to a$ and $b_n \to a$, for n large enough we have $[a_n,b_n] \subset (a-\epsilon,a+\epsilon) \subseteq U_{\alpha_0}$. These intervals $[a_n,b_n]$ can thus be covered by a finite subcover, namely the collection $\{U_{\alpha_0}\}$ consisting of only one member. This contradicts the construction of $[a_n,b_n]$. □

Remark 1.12. Any closed interval $[a,b] \subset \mathbb{R}$ is homeomorphic to $[0,1]$ and thus also compact.

Example 1.13. Consider the continuous map

$$f\colon [0,2\pi] \to S^1$$

$$t \mapsto (\cos t, \sin t)$$

which induces a continuous map on the quotient

$$\overline{f}\colon [0,2\pi]/\!\sim \to S^1$$

where the equivalence relation $\sim$ identifies the endpoints of the interval, i.e. is generated by $0 \sim 2\pi$. Then $\overline{f}$ is a continuous bijection, the domain $[0,2\pi]/\!\sim$ is compact, and S^1 is Hausdorff, therefore $\overline{f}$ is a homeomorphism.