Math 535 - General Topology
Additional notes

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1 Compactness and completeness in metric spaces

Definition 1.1. A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is a **Cauchy sequence** if for any $\epsilon > 0$, there is an index $N \in \mathbb{N}$ satisfying

$$d(x_m, x_n) < \epsilon$$

for all $m, n \geq N$.

Equivalently: $\lim_{n\to\infty} \sup_{k\in\mathbb{N}} d(x_n, x_{n+k}) = 0$.

Definition 1.2. A metric space (X, d) is **complete** if every Cauchy sequence in X converges.

Example 1.3. The real line $\mathbb{R}$ is complete, whereas the interval $(0, 1)$ is not complete.

Exercise 1.4. Let X be a complete metric space and $C \subseteq X$ a closed subset. Show that C is complete.

Slogan: "closed in complete is complete".

Definition 1.5. A metric space (X, d) is **totally bounded** if for every $\epsilon > 0$, X can be covered by finitely many ϵ-balls.

Theorem 1.6. *Let (X, d) be a metric space. Then the following are equivalent.*

1. *X is compact.*
2. *X is sequentially compact.*
3. *X is complete and totally bounded.*

Proposition 1.7 (Lebesgue covering lemma)**.** *Let (X, d) be a compact metric space and $\{U_\alpha\}_{\alpha\in A}$ an open cover of X. Then there is a number $\delta > 0$ such that for any $A \subseteq X$ with diam$(A) < \delta$, the inclusion $A \subseteq U_\alpha$ holds for some α.*

Such a number δ is called a ***Lebesgue number*** *of the cover.*

2 Uniform continuity

Definition 2.1. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is **uniformly continuous** if for any $\epsilon > 0$, there is a $\delta > 0$ satisfying

$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \epsilon.$$

Remark 2.2. In the definition of continuity, the $\delta = \delta(\epsilon, x)$ depends on ϵ and the point x, whereas uniform continuity means that the $\delta = \delta(\epsilon)$ does not depend on x.

In particular, a uniformly continuous map is always continuous, but not the other way around. For example, the map $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous but not uniformly continuous.

Proposition 2.3. *Let X and Y be metric spaces, where X is* compact*, and $f\colon X \to Y$ is continuous. Then f is uniformly continuous.*

Proof. Let $\epsilon > 0$ and consider the open cover $\{B_{\frac{\epsilon}{2}}(y)\}_{y\in Y}$ of Y. Taking preimages yields the open cover $\{f^{-1}B_{\frac{\epsilon}{2}}(y)\}_{y\in Y}$ of X. Since X is compact, this open cover has a Lebesgue number $\delta > 0$. The following implications hold:

$$\begin{aligned} d(x, x') < \delta &\Rightarrow x, x' \in f^{-1}B_{\frac{\epsilon}{2}}(y) \text{ for some } y \in Y \\ &\Rightarrow f(x), f(x') \in B_{\frac{\epsilon}{2}}(y) \\ &\Rightarrow d(f(x), f(x')) \leq d(f(x), y) + d(y, f(x')) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Definition 2.4. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is **Lipschitz continuous** if there is a constant $K \geq 0$ satisfying

$$d_Y(f(x), f(x')) \leq K d_X(x, x')$$

for all $x, x' \in X$

In other words, f distorts distances at most by a factor of K. Such a constant K is called a **Lipschitz constant** for f.

Proposition 2.5. *A* differentiable *function $f\colon (a, b) \to \mathbb{R}$ is Lipschitz continuous if and only if its derivative $f'\colon (a, b) \to \mathbb{R}$ is bounded. In that case, any Lipschitz constant is an upper bound on the absolute value of the derivative $|f'(x)|$, and vice versa.*

Proposition 2.6. *Lipschitz continuity implies uniform continuity.*

Proof. Take $\delta = \frac{\epsilon}{K}$. □

Example 2.7. The converse does not hold. For example, consider the function $f\colon [0, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Then f is uniformly continuous, since it is continuous and its domain $[0, 1]$ is compact. However f is not Lipschitz continuous, since the derivative $f'(x) = \frac{1}{2\sqrt{x}}$ goes to infinity as x goes to 0.