Math 535 - General Topology
Additional notes

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1 Stone-Čech compactification

Let X be a topological space and consider the set of all continuous functions on X with values in $[0, 1]$

$$C := \{f : X \to [0, 1] \mid f \text{ is continuous}\}.$$

Consider the set $[0, 1]^C \cong \prod_{f \in C} [0, 1]$ of all functions from C to $[0, 1]$, endowed with the product topology. Consider the evaluation map

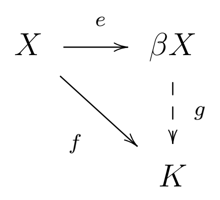
$$\begin{aligned} e : X &\to [0, 1]^C \\ x &\mapsto (f(x))_{f \in C}. \end{aligned}$$

so that $e(x)$ is "evaluation at x".

Definition 1.1. The **Stone-Čech construction** on X is the space $\beta X := \overline{e(X)}$ together with the map $e : X \to \beta X$.

Note that $e(X)$ is dense in βX, and βX is always compact Hausdorff. The map $e : X \to \beta X$ is an embedding if and only if X is Tychonoff ($T_{3\frac{1}{2}}$), in which case $e : X \hookrightarrow \beta X$ is a compactification of X, called the **Stone-Čech compactification** of X.

Theorem 1.2. *Let X be a topological space. Then the Stone-Čech construction $e : X \to \beta X$ satisfies the following universal property. For any compact Hausdorff space K and any continuous map $f : X \to K$, there is a unique continuous map $g : \beta X \to K$ satisfying $g \circ e = f$, that is, making the diagram*



commute.

Remark 1.3. As usual, because $e : X \to \beta X$ satisfies a universal property, it is unique up to unique isomorphism. More precisely, if $e' : X \to Z$ is a continuous map to a compact Hausdorff space Z thast satisfies the universal property of theorem 1.2, then there is a unique homeomorphism $h\colon \beta X \xrightarrow{\cong} Z$ which is compatible with the evaluation maps, meaning $h \circ e = e'$, i.e. making the diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & \beta X \\ & {\scriptstyle e'}\searrow & \cong \downarrow h \\ & & Z \end{array}$$

commute.

"The" Stone-Čech construction may as well refer to any such $e'\colon X \to Z$. Definition 1.1 provided one specific construction which works.

Corollary 1.4. *Let X be a Tychonoff space, and $f\colon X \to \mathbb{R}$ a* bounded *continuous function. Then f admits a unique continuous extension to βX.*

Remark 1.5. The extension property in 1.4 characterizes the Stone-Čech compactification. If $e'\colon X \to Z$ is a Hausdorff compactification of X that satisfies the extension property for bounded real-valued functions, then $e'\colon X \to Z$ is the Stone-Čech compactification of X.