# Math 535 - General Topology
# Additional notes

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## 1 Compactly generated spaces

**Definition 1.1.** A topological space $X$ is **compactly generated** if the following holds: a subset $A \subseteq X$ is open in $X$ if and only if $A \cap K$ is open in $K$ for every compact subset $K \subseteq X$.

Equivalently, $A \subseteq X$ is closed in $X$ if and only if $A \cap K$ is closed in $K$ for every compact subset $K \subseteq X$.

*Example* 1.2. Any compact space is compactly generated.

**Proposition 1.3.** *A topological space $X$ is compactly generated if and only if the following holds: a subset $A \subseteq X$ is open in $X$ if and only if for every compact space $K$ and continuous map $f\colon K \to X$, the preimage $f^{-1}(A)$ is open in $K$.*

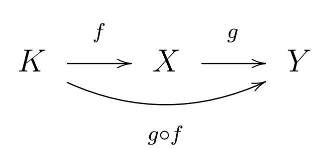
*Remark* 1.4. Some authors have a slightly different definition of compactly generated, possibly imposing the Hausdorff condition (or a weaker separation axiom) to $X$ or to $K$.

Some authors call definition 1.1 **k-space**, while some reserve the term k-space for a slightly different notion.

In the definition 1.1 above, a subset $A \subseteq X$ such that $A \cap K$ is open in $K$ for every compact subset $K \subseteq X$ deserves to be called **k-open** in $X$. Every open in $X$ is k-open, and $X$ being compactly generated means that every k-open in $X$ is open in $X$.

**Proposition 1.5.** *Let $X$ be a compactly generated space, $Y$ a topological space, and $g\colon X \to Y$ a map (not necessarily continuous). The following are equivalent.*

1. *$g\colon X \to Y$ is continuous.*
2. *For all compact subset $K \subseteq X$, the restriction $g|_K\colon K \to Y$ is continuous.*
3. *For all compact space $K$ and continuous map $f\colon K \to X$, the composite $g \circ f\colon K \to Y$ is continuous.*



Many spaces are compactly generated.

**Proposition 1.6.** *Any first-countable space is compactly generated.*

**Proposition 1.7.** *Any locally compact space is compactly generated. [Locally compact in the weak sense, i.e. every point has a compact neighborhood.]*

However, not all spaces are compactly generated.

**Proposition 1.8.** *An uncountable product of copies of $\mathbb{R}$ is* not *compactly generated.*

## 2 k-ification

A space $X$ may not be compactly generated, but we now describe the "best approximation" of $X$ by a compactly generated space.

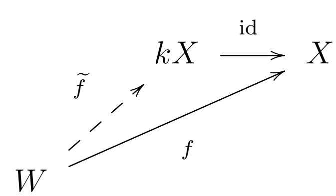
**Definition 2.1.** Let $(X, \mathcal{T})$ be a topological space. The collection $\mathcal{T}_{cg}$ of k-open subsets of $X$ (i.e. subsets $A \subseteq X$ such that $A \cap K$ is open in $K$ for any compact subset $K \subseteq X$) is a topology on $X$. The **k-ification** of $X$ is the topological space $kX := (X, \mathcal{T}_{cg})$.

Since open subsets of $X$ are always k-open in $X$, the inclusion of topologies $\mathcal{T} \subseteq \mathcal{T}_{cg}$ always holds, i.e. the identity function $\mathrm{id}\colon kX \to X$ is continuous. [Here id is an abuse of notation, since $kX$ and $X$ are usually different topological spaces.]

The continuous map $\mathrm{id}\colon kX \to X$ satisfies the universal property described in the following proposition.

**Proposition 2.2.** *Let $X$ be a topological space.*

1. *The k-ification $kX$ is compactly generated.*
2. *For any compactly generated space $W$ and continuous map $f\colon W \to X$, there exists a unique continuous map $\widetilde{f}\colon W \to kX$ satisfying $f = \mathrm{id} \circ \widetilde{f}$, i.e. making the diagram*

   *commute.*

   *Note that $\widetilde{f}$ has the same underlying function as $f\colon W \to X$, i.e. $\widetilde{f}(w) = f(w)$ for all $w \in W$. The claim is that this function is continuous when viewed as a map $W \to kX$.*

*Proof.* Homework 14 Problem 3. □

This means that $kX \to X$ is the "closest" compactly generated space that maps into $X$.

We can now prove a converse to proposition 1.5.

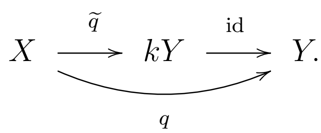
**Proposition 2.3.** *Let $X$ be a topological space such that "compact subsets detect continuity" in the following sense: For any topological $Y$ and map $g\colon X \to Y$, $g$ is continuous whenever its restriction $g|_K\colon K \to Y$ to any compact subset $K \subseteq X$ is continuous. Then $X$ is compactly generated.*

*Proof.* Let $K \subseteq X$ be a compact subset. Since $K$ is in particular compactly generated, the inclusion $K \hookrightarrow X$ induces a continuous map $K \hookrightarrow kX$.

Now consider the identity function $\mathrm{id}\colon X \to kX$. For any compact subset $K \subseteq X$, the restriction of id to $K$ is the inclusion map $\mathrm{id}|_K\colon K \hookrightarrow kX$, which is continuous, as noted above. Therefore the assumption on $X$ implies that $\mathrm{id}\colon X \to kX$ is continuous, so that $kX \cong X$ is a homeomorphism and $X$ is compactly generated. □

**Proposition 2.4.** *Any quotient of a compactly generated space is compactly generated.*

*Proof.* Let $X$ be a compactly generated space, and $q\colon X \twoheadrightarrow Y$ a quotient map, which is in particular continuous. Since $X$ is compactly generated, $q$ induces a continuous map $\widetilde{q}\colon X \to kY$, as illustrated here:




Let $A \subseteq Y$ be a k-open subset of $Y$, which means $\mathrm{id}^{-1}(A) \subseteq kY$ is open in $kY$. Then the preimage

$$\begin{aligned} q^{-1}(A) &= (\mathrm{id} \circ \widetilde{q})^{-1}(A) \\ &= \widetilde{q}^{-1}\left(\mathrm{id}^{-1}(A)\right) \subseteq X \end{aligned}$$

is open in $X$ since $\widetilde{q}\colon X \to kY$ is continuous. Therefore $A \subseteq Y$ is open in $Y$ since $q$ is a quotient map. □

We can now prove a structure theorem for compactly generated spaces.

**Proposition 2.5.** *A topological space is compactly generated if and only if it is a quotient of a locally compact space.*

*Proof.* ($\Leftarrow$) A locally compact space is always compactly generated (by 1.7). Therefore a quotient of a locally compact space is also compactly generated (by 2.4).

($\Rightarrow$) A compactly generated space is a quotient of a coproduct $\coprod_{i\in I} K_i$ of compact spaces $K_i$, by Homework 14 Problem 4.

Moreover, a coproduct of compact spaces is locally compact. Indeed, every point $w \in \coprod_{i\in I} K_i$ lives in a summand $K_j$, and $K_j$ is a compact neighborhood of $w \in K_j$, since $K_j$ is open in the coproduct $\coprod_{i\in I} K_i$. □