Math 535 - General Topology
Fall 2012
Homework 10, Lecture 11/2

Problem 5. Two objects X and Y of a category $\mathcal{C}$ are **connected by morphisms** if there is a zigzag of morphisms between them. More precisely, there is a finite sequence of objects

$$X = X_0, X_1, \ldots, X_{n-1}, X_n = Y$$

and for every $0 \leq i < n$, there is a morphism $f_i \colon X_i \to X_{i+1}$ or $f_i \colon X_{i+1} \to X_i$.

a. Show that two objects X and Y of a groupoid $\mathcal{G}$ are connected by morphisms if and only if there is a morphism $f \colon X \to Y$.

Remark. In particular, two points x and y in a space X are connected by morphisms in the fundamental groupoid $\Pi_1(X)$ if and only if they lie in the same path component of X.

b. Find an example of category $\mathcal{C}$ and objects X and Y of $\mathcal{C}$ that are connected by morphisms, but such that there are no morphisms from X to Y and no morphisms from Y to X:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \emptyset \text{ and } \mathrm{Hom}_{\mathcal{C}}(Y, X) = \emptyset.$$

c. Let X and Y be objects of a groupoid $\mathcal{G}$ that are connected by morphisms. Show that the vertex groups at X and Y are isomorphic (as groups):

$$\mathrm{Aut}_{\mathcal{G}}(X) \simeq \mathrm{Aut}_{\mathcal{G}}(Y).$$

Remark. This proves in particular that if x and y are two points in the same path component of a space X, then the fundamental groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic.

Problem 6. Let $f \colon X \xrightarrow{\simeq} Y$ be a homotopy equivalence between topological spaces. Show that for any choice of basepoint $x_0 \in X$, the induced group homomorphism

$$\pi_1(f) \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.