Math 535 - General Topology
Fall 2012
Homework 11, Lecture 11/7

Definition. Let X be a topological space. A function $f\colon X \to \mathbb{R}$ is **lower semicontinuous** if for all $a \in \mathbb{R}$, the preimage $f^{-1}(a, +\infty)$ is open in X.

Equivalently: For all $x_0 \in X$ and $\epsilon > 0$, there is a neighborhood U of x_0 satisfying $f(x) > f(x_0) - \epsilon$ for all $x \in U$. This means that the values close to x_0 can "suddenly jump up" but not down.

Problem 2 again.

b. Let X be a Baire space and $f\colon X \to \mathbb{R}$ a lower semicontinuous function. Show that for every non-empty open subset $U \subseteq X$, there is a non-empty open subset $V \subseteq U$ on which f is bounded above.

Problem 3. Show that a topological space X is of second category in itself if and only if any countable intersection of open dense subsets of X is non-empty.

Problem 4. (Uniform boundedness principle) (Willard Exercise 25D.5) (Munkres Exercise 48.10) (Bredon I.17.2)

Let X be a Baire space and $S \subseteq C(X, \mathbb{R})$ a collection of real-valued continuous functions on X which is pointwise bounded: for each $x \in X$, there is a bound $M_x \in \mathbb{R}$ satisfying

$$|f(x)| \leq M_x \text{ for all } f \in S.$$

Show that there is a non-empty open subset $U \subseteq X$ on which the collection S is uniformly bounded: there is a bound $M \in \mathbb{R}$ satisfying

$$|f(x)| \leq M \text{ for all } x \in U \text{ and all } f \in S.$$