Math 535 - General Topology
Fall 2012
Homework 13, Lecture 11/26

Problem 1. Let X be a compact topological space, and (Y, d) a metric space. Consider the uniform metric

$$d(f, g) := \sup_{x \in X} d(f(x), g(x))$$

on the set of continuous maps $C(X, Y)$.

Show that the topology on $C(X, Y)$ induced by the uniform metric is the compact-open topology.

Problem 2. Let X and Y be topological spaces. Let $f, g \colon X \to Y$ be two continuous maps. Show that a homotopy from f to g induces a (continuous) path from f to g in the space of continuous maps $C(X, Y)$ endowed with the compact-open topology.

More precisely, let $F(X, Y)$ denote the *set of all functions* from X to Y. There is a natural bijection of sets:

$$\varphi \colon F(X \times [0,1], Y) \xrightarrow{\cong} F([0,1], F(X, Y))$$

sending a function $H \colon X \times [0,1] \to Y$ to the function $\varphi(H) \colon [0,1] \to F(X, Y)$ defined by $\varphi(H)(t) = H(-, t) =: h_t$.

Your task is to show that if a function $H \colon X \times [0,1] \to Y$ is continuous, then the following two conditions hold:

1. $h_t \colon X \to Y$ is continuous for all $t \in [0,1]$;
2. The corresponding function $\varphi(H) \colon [0,1] \to C(X, Y)$ is continuous.

Remark. If X is *locally compact Hausdorff*, then the converse holds as well: the two conditions guarantee that $H \colon X \times [0,1] \to Y$ is continuous. In that case, a homotopy from f to g is really the same as a path from f to g in the function space $C(X, Y)$.