Math 535 - General Topology
Fall 2012
Homework 14, Lecture 12/3

Problem 1. Let X be a topological space and (Y, d) a metric space. For each compact subset $K \subseteq X$, consider the pseudometric on $C(X, Y)$ defined by

$$d_K(f, g) = \sup_{x \in K} d\left(f(x), g(x)\right)$$

and its associated open balls $B_K(f, \epsilon) = \{g \in C(X, Y) \mid d_K(f, g) < \epsilon\}$.
Show that the collection of all open balls

$$\mathcal{B} = \{B_K(f, \epsilon) \mid K \subseteq X \text{ compact}, f \in C(X, Y), \epsilon > 0\}$$

forms a basis for a topology on $C(X, Y)$. More explicitly:

1. $\mathcal{B}$ covers $C(X, Y)$;
2. Finite intersections of members of $\mathcal{B}$ are unions of members of $\mathcal{B}$.

The following proposition will be relevant to Problem 2. **Do not** prove the proposition in your write-up.

Proposition. *1. Given a pseudometric d on a set X, there is a topologically equivalent pseudometric ρ on X which is bounded above by 1.*
For example, the formulas $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ or $\rho(x, y) = \min\{d(x, y), 1\}$ work.

2. Given a countable family of pseudometrics $\{d_n\}_{n \in \mathbb{N}}$ on X which are bounded above by 1, the formula

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x, y) \tag{1}$$

defines a pseudometric d on X.

3. The topology $\mathcal{T}_d$ on X induced by d is the topology generated by $\bigcup_{n \in \mathbb{N}} \mathcal{T}_{d_n}$. More explicitly, this is the topology generated by the collection of all open balls

$$\{B_n(x, \epsilon) \mid n \in \mathbb{N}, x \in X, \epsilon > 0\}$$

where we used the notation $B_n(x, \epsilon) := \{y \in X \mid d_n(x, y) < \epsilon\}$.

Proof. Essentially Homework 6 Problem 4.

1. Parts (a) and (b).
2. Part (c).
3. Slight generalization of part (d).

□

Problem 2. A family of pseudometrics $\{d_\alpha\}_{\alpha \in A}$ on a set X is **separating** if the following implication holds:

$$d_\alpha(x, y) = 0 \text{ for all } \alpha \in A \Rightarrow x = y.$$

In other words, for any distinct points $x \neq y$, there is an index $\alpha \in A$ satyisfying $d_\alpha(x, y) > 0$.

a. Let X be a set and $\{d_n\}_{n \in \mathbb{N}}$ a countable family of pseudometrics on X which are bounded above by 1. Let d be the pseudometric on X defined by the formula (1) as in the proposition above.

Show that d is a metric if and only if the family of pseudometrics $\{d_n\}_{n \in \mathbb{N}}$ is separating.

b. Let (Y, d) be a metric space and consider the mapping space $C(\mathbb{R}, Y)$. For all $n \in \mathbb{N}$, consider the compact interval $[-n, n] \subset \mathbb{R}$ and the associated pseudometric

$$d_n(f, g) = \sup_{x \in [-n,n]} d\left(f(x), g(x)\right).$$

Show that the family of pseudometrics $\{d_n\}_{n \in \mathbb{N}}$ on $C(\mathbb{R}, Y)$ is separating.

c. Show that the topology $\mathcal{T}$ on $C(\mathbb{R}, Y)$ generated by $\bigcup_{n \in \mathbb{N}} \mathcal{T}_{d_n}$ is the topology of compact convergence.