Math 535 - General Topology
Fall 2012
Homework 1, Lecture 8/27

Definition. Let V be a (real or complex) vector space. A **norm** on V is a function $\|\cdot\|: V \to \mathbb{R}$ satisfying:

1. Positivity: $\|x\| \geq 0$ for all $x \in V$ and moreover $\|x\| = 0$ holds if and only if $x = 0$.
2. Homogeneity: $\|\alpha x\| = |\alpha| \|x\|$ for any scalar α and $x \in V$.
3. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

A **normed vector space** is the data $(V, \|\cdot\|)$ of a vector space V equipped with a norm $\|\cdot\|$.

Problem 1. Let $(V, \|\cdot\|)$ be a normed vector space. Define a function $d: V \times V \to \mathbb{R}$ by

$$d(x, y) := \|x - y\|.$$

Show that d is a metric on V, called the metric **induced** by the norm $\|\cdot\|$.

Problem 2. Denote by $\|\cdot\|_2$ the standard (Euclidean) norm on $\mathbb{R}^n$, defined by

$$\|x\|_2 := \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}.$$

Now consider the function $\|\cdot\|_1: \mathbb{R}^n \to \mathbb{R}$ defined by

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|.$$

a. Show that $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$.

Remark. The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are special cases of the so-called p-norm, for any real number $p \geq 1$ or $p = \infty$. See:

`http://en.wikipedia.org/wiki/Lp_spaces#The_p-norm_in_finite_dimensions`.

b. Find constants $C, D > 0$ satisfying

$$\|x\|_2 \leq C \|x\|_1$$

$$\|x\|_1 \leq D \|x\|_2$$

for all $x \in \mathbb{R}^n$.

Definition. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V are **equivalent** if they can be compared as in Problem 2b.

Definition. Two metrics d_1 and d_2 on a set X are **topologically equivalent** if for every $x \in X$ and $\epsilon > 0$, there is a $\delta > 0$ satisfying

$$d_1(x, y) < \delta \Rightarrow d_2(x, y) < \epsilon$$

$$d_2(x, y) < \delta \Rightarrow d_1(x, y) < \epsilon.$$

In other words, the identity function $(X, d_1) \to (X, d_2)$ is a homeomorphism.

Problem 3. Show that equivalent norms on a vector space V induce topologically equivalent metrics.

Problem 4. (Bredon Prop. I.1.3) Show that topologically equivalent metrics induce the same topology (which explains the terminology). In other words, if d_1 and d_2 are topologically equivalent metrics on X, then a subset $U \subseteq X$ is open with respect to d_1 if and only if it is open with respect to d_2.