Math 535 - General Topology
Fall 2012
Homework 1, Lecture 8/31

Problem 8. Let $f\colon X \to Y$ be a function between topological spaces, and let $x \in X$.

a. Show that the following conditions (defining continuity of f at x) are equivalent.

1. For all neighborhood N of $f(x)$, there is a neighborhood M of x such that $f(M) \subseteq N$.
2. For all open neighborhood V of $f(x)$, there is an open neighborhood U of x such that $f(U) \subseteq V$.
3. For all neighborhood N of $f(x)$, the preimage $f^{-1}(N)$ is a neighborhood of x.

b. Find an example of function $f\colon X \to Y$ between *metric* spaces which is continuous at a point $x \in X$, but there is an open neighborhood V of $f(x)$ such that the preimage $f^{-1}(V)$ is *not* an open neighborhood of x.

Upshot: The description "preimage of open is open" is really about global continuity, not pointwise continuity (or even local continuity).

Problem 9. Let X be a topological space and $\mathcal{B}$ a collection of open subsets of X.

a. Show that $\mathcal{B}$ is a basis for the topology of X if and only if for every open subset $U \subseteq X$ and $x \in U$, there is a $B \in \mathcal{B}$ satisfying $x \in B \subseteq U$.

b. Assuming X is a metric space, show that the collection of open balls

$$\mathcal{B} = \{B_{\frac{1}{n}}(x) \mid x \in X, n \in \mathbb{N}\}$$

is a basis for the topology of X.

Problem 10. Let X be a set and $\mathcal{S}$ a collection of subsets of X.

a. Show that the collection

$$\mathcal{T} := \left\{ \bigcup_{\alpha} \bigcap_{i=1}^{n_\alpha} S_{\alpha,i} \mid S_{\alpha,i} \in \mathcal{S} \right\}$$

of (arbitrary) unions of finite intersections of members of $\mathcal{S}$ is a topology on X.

b. Show that $\mathcal{T}$ is the topology $\mathcal{T}_{\mathcal{S}}$ generated by $\mathcal{S}$. In other words: $\mathcal{T}$ contains $\mathcal{S}$ and any other topology $\mathcal{T}'$ containing $\mathcal{S}$ must satisfy $\mathcal{T} \leq \mathcal{T}'$.