Math 535 - General Topology
Fall 2012
Homework 3, Lecture 9/14

Note: Problem 7 from Lecture 9/12 really belongs here.

Problem 8. Show that a countable product of first-countable topological spaces is first-countable. In other words, if the spaces $X_1, X_2, X_3, \ldots$ are first-countable, then their product $\prod_{i \in \mathbb{N}} X_i$ (with the product topology) is also first-countable.

Problem 9. Let X be a topological space. A subset $A \subseteq X$ is called **dense** in X if its closure is all of X, i.e. $\overline{A} = X$.

Show that A is dense in X if and only if every non-empty open subset of X contains a point of A.

Problem 10. A topological space X is called **separable** if it contains a countable dense subset.

a. Show that a second-countable space is always separable.

Now we will show that the converse statement does not hold.

b. Let X be an *uncountable* set (e.g. the real numbers $\mathbb{R}$) endowed with the *cofinite* topology. Show that X is separable.

c. Show that X from part (b) is not first-countable (let alone second-countable).